2017/18

Please refer to the University website for further details of the current Coursework Regulations, including details of penalties for late submission, procedures for Extenuating Circumstances, and penalties for Assessment Offences. See **http://www2.gre.ac.uk/current-students/regs**.

- **Detailed Specification**

  This coursework is to be completed individually.

- **Grading and Assessment Criteria**

  The total possible marks for this assessment is 100, and represents the total possible marks for this course. The marks for each part of a question are indicated, and the total marks for each question are indicated at the end of the question.

Mathematics for the Life Sciences (MATH1134) Coursework

Mathematics for the Life Sciences Coursework

There are 6 questions. **You must answer all questions** for full marks. Any handwritten work should be scanned so that you can include it as an image in the single PDF file you upload.

You should provide a listing of any MATLAB script and function M-files used to answer the questions (except for the M-files provided on the course Moodle page or any in-built MATLAB functions that you use, in which case you should clearly state what M-file(s) or in-built function(s) you used to answer the questions). Any MATLAB M-file which you include should **not** be included as an image or as images. Instead, it is recommended that you use Word when possible so that you can easily copy and paste the content of your MATLAB M-files, data and figures, and then use its "Save As a PDF File" or "Export As a PDF File" option. You are free to edit and adapt any of the MATLAB M-files posted on the Moodle site for MATH1134. If you choose to use Python, all of the instructions above also apply with the relevant substitution of "Python" for "MATLAB".

This coursework is divided into three sections. Every question in each section is compulsory:

Section A *(Questions 1–3)* This section primarily assesses your analytical skills; your ability to apply the mathematics you have learned to specific examples of biological systems.

Section B *(Questions 4–5)* This section primarily assesses your research and expository skills.

Section C *(Question 6)* This section primarily assesses your understanding of the context in which mathematics is applied to the life sciences.

IF THERE IS ANY ASPECT OF THE PRECEDING OR FOLLOWING INSTRUCTIONS WHICH YOU DO NOT UNDERSTAND, PLEASE CONSULT THE COURSE LECTURER(S) FOR CLARIFICATION.

Section A

1. The sizes of two interacting populations, $x_1(t)$ and $x_2(t)$, are modelled over a short time frame by the linear system of differential equations

$$\vec{x}\,' = A\vec{x}, \quad \text{where } A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Solve this system of differential equations along with the initial conditions

$$\vec{x}(0) = \begin{bmatrix} 1500 \\ 1000 \end{bmatrix}.$$

[5 MARKS TOTAL]

2. A model for population growth is given by

$$\frac{dN}{dt} \quad = \quad f(N) \quad = \quad rN\left(1 - a(N-b)^2\right)$$

where r, a, and b are positive parameters.

(a) Find all equilibrium solutions of this equation and use calculus to classify each as asymptotically stable, semistable, or unstable. Where necessary, this classification should involve indicating *ranges* of parameter values for which the equilibrium solution, N^*, is asymptotically stable, semistable, or unstable, but for simplicity consider only the cases where $f\,'(N^*) < 0$ and $f\,'(N^*) > 0$, **EXCEPT for the equilibrium solution, N_0, with the SMALLEST MAGNITUDE - in which case you should also discuss stability when $f\,'(N_0) = 0$.**

[8 marks]

(b) Confirm your classification of the equilibrium solutions in the special case $r = 1$, $a = \frac{1}{4}$, and $b = 4$ by producing a suitable direction field plot which shows the equilibrium solutions. Clearly state what the equilibrium solutions are and classify them using the direction field, stating clearly what aspect of the direction field allows you to conclude whether an equilibrium solution is asymptotically stable, semistable, or unstable.

[4 marks]

(c) Let $r = 1$ and choose values of a and b such that $ab^2 = 1$. Use these values to produce a plot of $f(N)$ versus N which shows all of the equilibrium solutions. Classify each equilibrium solution using only this $f(N)$ versus N plot, clearly stating what feature(s) of your plot allows you to conclude whether an equilibrium solution is asymptotically stable, semistable, or unstable.

[4 marks]

(d) Show that this model exhibits the *Allee effect* (HINT: you can do this just by looking at the differential equation). For the parameter values given in part (b), say with reason(s) whether the model exhibits a **strong** or **weak** *Allee effect*.

Note: to answer this question you will have to do some additional research on the *Allee effect* beyond the brief introduction to the phenomenon given in **Lecture 3**. Be sure to cite any reference(s) used.

[4 marks]

QUESTION 2 CONTINUES ON THE NEXT PAGE

(e) Consider the following slight modification to the differential equation given earlier in this question:

$$\frac{dN}{dt} \quad = \quad h(N) \quad = \quad rN - aN(N-b)^2.$$

where again r, a, and b are positive parameters. Follow the procedure outlined in **Lecture 5** to express the differential equation above in non-dimensional terms so that there remains only one dimensionless parameter $k = b\left(\frac{a}{r}\right)^{\frac{1}{2}}$ and the equation becomes

$$\frac{du}{d\tau} \quad = \quad u\left(1 - [u-k]^2\right)$$

where N has been replaced with a non-dimensional u and t replaced with a non-dimensional τ. State clearly what the new dimensionless variables u and τ are in terms of the old variables and parameters.

[5 marks]

[25 MARKS TOTAL]

3. In a type of interaction known as *mutualism*, it has been observed that in some cases each species in a set of different species benefits from its association with the other species. In 1976 in his book, *Theoretical Ecology, Principles and Applications*, Robert May suggested the following model for mutualism between two species:

$$\frac{dN_1}{dt} = rN_1\left(1 - \frac{N_1}{K_1 + \alpha N_2}\right), \qquad \frac{dN_2}{dt} = rN_2\left(1 - \frac{N_2}{K_2 + \beta N_1}\right)$$

where $N_i(t)$ is the population of species i at time t, $r > 0$ is the intrinsic growth rate of both species, and K_1, K_2, α, and β are positive constants such that $\alpha < \frac{1}{\beta}$. Note that in general r need not be the same for both species, but we we make that assumption here to simplify some of the calculations required in this question.

(a) Find all steady states of this system of equations and classify them using calculus and linear algebra.

[9 marks]

(b) Choose an appropriate set of parameters and produce an appropriate direction field plot on which you identify specific cases of all of the equilibrium solutions discovered in part (a). Be sure to list the equilibrium solutions for the specific set of parameter values you have chosen. Note this plot should confirm your classification of the equilibrium solutions from part (a).

[3 marks]

(c) Explain why the equations describe mutualistic behaviour.

[2 marks]

(d) What is the reason for the requirement that $\alpha < \frac{1}{\beta}$ if this system of differential equations is to model coexistence of two species in a mutualistic relationship?

[2 marks]

(e) Solve this system of differential equations using at least a second order numerical method (so Heun's method or Matlab's in-built `ode45` are acceptable but not Euler's method) with $r = 0.5$, $K_1 = 10000$, $K_2 = 15000$, $\alpha = 0.7$, $\beta = 0.8$ over the interval $t \in [0, 100]$. Use two sets of different (non-zero) starting population vectors $\vec{N_0}$ which produce qualitatively different solution behaviours (such as increasing graphs versus decreasing graphs), show the plots of those solutions, and discuss what is happening to the two populations in those two cases.

[4 marks]

[20 MARKS TOTAL]

Section B

4. Write a mini-lecture on the Gompertz equation which is appropriate for a final year undergraduate student in Mathematics. This mini-lecture should include a description of at least two examples of scenarios the equation is used to model and why the equation is used in those scenarios. A discussion of equilibrium solutions and a derivation of the solution to the Gompertz equation should be included, as well as a test example showing the use of the equation in the modelling of a specific scenario. You may either use the exact solution or a numerical approximation to the solution to demonstrate the use of the Gompertz equation in this modelling scenario. As a guideline, your mini-lecture could include some or all of the following: Historical/background information on the equation; some modelling uses of the equation (overview); derivation of the equation; description and analysis of the qualitative behaviour of solutions to the equation; the analytical solution of the equation; an application of the equation in a specific modelling situation.

   Cite all sources of information using a Harvard referencing style and do not rely solely on Wikipedia for your information. **The guide length for this mini-lecture is 1000 words**.

   Marks for this question will be allocated as follows:

   - Content: 8 marks.
   - Writing Style *(including readability, grammar, punctuation, spelling)*: 5 marks.
   - Referencing: 2 marks.

   A first class mini-lecture would adequately address all requirements listed, be logically structured and easy to follow, use appropriate referencing, and have few if any grammatical and spelling errors. A second class mini-lecture might not adequately address one or two of the requirements listed, and/or might suffer from poor structuring, and/or might not contain an appropriate level of information, and/or might have limited or incorrect referencing, and/or might contain a significant number of grammatical and spelling errors. A third class mini-lecture might not adequately address several of the requirements listed, and/or might suffer from poor structuring, and/or might not contain an appropriate level of information, and/or might have very poor referencing, and/or might contain a significant number of grammatical and spelling errors. A failing mini-lecture would have limited useful information, likely poor structure, and/or possibly no or very poor referencing, and/or be replete with grammatical and spelling errors.

[15 MARKS TOTAL]

5. Write a mini-lecture on a mathematical biology model which uses a system of differential equations and which was not covered in class (or previously in this coursework). This mini-lecture should be appropriate for a final year Mathematics undergraduate student and should contain some biological and/or mathematical and/or historical background, a derivation of the model if feasible and/or an explanation of the logic behind the model, and a qualitative description of solutions including a discussion of steady states.

   Cite all sources of information using a Harvard referencing style and do not rely solely on Wikipedia for your information. **The guide length for this mini-lecture is 1000 words**.

   Marks for this question will be allocated as follows:

   - Content: 8 marks.
   - Writing Style *(including readability, grammar, punctuation, spelling)*: 5 marks.
   - Referencing: 2 marks.

   A first class mini-lecture would adequately address all requirements listed, be logically structured and easy to follow, use appropriate referencing, and have few if any grammatical and spelling errors. A second class mini-lecture might not adequately address one or two of the requirements listed, and/or might suffer from poor structuring, and/or might not contain an appropriate level of information, and/or might have limited or incorrect referencing, and/or might contain a significant number of grammatical and spelling errors. A third class mini-lecture might not adequately address several of the requirements listed, and/or might suffer from poor structuring, and/or might not contain an appropriate level of information, and/or might have very poor referencing, and/or might contain a significant number of grammatical and spelling errors. A failing mini-lecture would have limited useful information, likely poor structure, and/or possibly a choice of topics which is inconsistent with the question asked, and/or possibly no or very poor referencing, and/or be replete with grammatical and spelling errors.

   [15 MARKS TOTAL]

Section C

6. Ian Stewart writes in *The Mathematics of Life* that "for the next century the driving force behind mathematics will be biology, and that this marks a fundamental, and exciting, shift in how the sciences inter-relate". Discuss the reasons why biological applications are becoming important to mathematicians in the first quarter of the 21$^{\text{st}}$ century in a way which was not the case in the nineteenth and twentieth centuries. Your discussion should be targeted at students who have just completed A-level Mathematics or are first year University undergraduates studying Mathematics.

   You must cite all sources using the **Harvard referencing style**. The guide length for the essay is 1000 words.

   For this question marks will be awarded for:

   - accuracy of analysis;
   - quality of research (including suitability of your sources);
   - understanding of the issues;
   - appropriateness of presentation; and
   - quality of referencing.

   A first class answer (15 marks or more) would require excellent research, accurate analysis, and clear presentation. A second class answer (11 to 14 marks) might show some misunderstanding, some analysis not supported by evidence, or a presentation too technical for the intended audience. An answer showing inadequate research, weak analysis, or a significant lack of understanding will be awarded 8 marks or fewer.

   [20 MARKS TOTAL]