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A NOTE ON FINITE GROUPS IN WHICH ALL NONCYCLIC PROPER SUBGROUPS HAVE THE SAME ORDER[1]

Jiangtao Shi and Cui Zhang

School of Mathematics and Information Sciences, Yantai University, Yantai 264005, *China*
e-mails: shijt@pku.org.cn; cuizhang2008@gmail.com

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In this note, we give a complete classification of finite groups in which all noncyclic proper subgroups have the same order.

Key words : Finite group; noncyclic proper subgroup; the same order.

1. Introduction

Throughout this note all groups are assumed to be finite. It is known that a group G is said to be cyclic if G can be generated by only one element. And a group G is said to be a minimal noncyclic group if G is noncyclic but all proper subgroups of G are cyclic. In [3], Miller and Moreno classified minimal noncyclic groups. They had:

***Theorem* 1** [3] — *A group G is a minimal noncyclic group if and only if G is one of the following groups:*

(1) $\mathbb{Z}_p \times \mathbb{Z}_p$*, where p is a prime;*

(2) Q_8*, the quaternion group of order* 8*;*

(3) $\langle a, b \mid a^p = b^{q^m} = 1, b^{-1}ab = a^r \rangle$*, where p and q are distinct primes and* $r \not\equiv 1 \,(\mathrm{mod}\, p)$*,* $r^q \equiv 1 \,(\mathrm{mod}\, p)$.

As a generalization of [3], Li and Zhao in [1] classified groups in which all noncyclic proper subgroups have exactly one conjugacy class.

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***Theorem* 2** [*1, Theorem 3.1*] — *A group G is a group in which all noncyclic proper subgroups have exactly one conjugacy class if and only if one of the following statements holds:*

(1) $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ *for some prime p;*

(2) $G = \langle a, b \mid a^{p^2} = b^p = 1,\ b^{-1}ab = a^{1+p}\rangle$ *for some odd prime p;*

(3) $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$*, where p and q are distinct primes;*

(4) $G = Q_8 \times \mathbb{Z}_p$ *for some odd prime p;*

(5) $G = H \times \mathbb{Z}_t$*, where* $H = \langle a, b \mid a^p = b^{q^m} = 1,\ b^{-1}ab = a^r\rangle$*, where p, q and t are distinct primes and* $r \not\equiv 1 \,(\mathrm{mod}\, p)$*,* $r^q \equiv 1 \,(\mathrm{mod}\, p)$*;*

(6) $G = \langle a, b \mid a^{p^2} = b^{q^m} = 1,\ b^{-1}ab = a^r\rangle$*, where p and q are distinct primes and* $r \not\equiv 1 \,(\mathrm{mod}\, p^2)$*,* $r^q \equiv 1 \,(\mathrm{mod}\, p^2)$*;*

(7) $G = {\mathbb{Z}_p}^2 \rtimes \mathbb{Z}_q$ *and* $[{\mathbb{Z}_p}^2, \mathbb{Z}_q] = {\mathbb{Z}_p}^2$*, where p and q are distinct primes and* $q \nmid p - 1$*;*

(8) $G = Q_8 \rtimes \mathbb{Z}_3$ *and* $[Q_8, \mathbb{Z}_3] = Q_8$*;*

(9) $G = \langle a, b \mid a^p = b^{q^m} = 1,\ b^{-1}ab = a^r\rangle$*, where p and q are distinct primes,* $m \geq 2$ *and* $r^q \not\equiv 1 \,(\mathrm{mod}\, p)$*,* $r^{q^2} \equiv 1 \,(\mathrm{mod}\, p)$*.*

As a generalization of [1, Theorem 3.1], Meng *et al.* [2] classified groups in which all noncyclic proper subgroups have exactly two conjugacy classes.

It is obvious that any two conjugate subgroups must have the same order but any two subgroups of the same order might not be conjugate. In this note, as a further generalization of [1, Theorem 3.1], we will give a complete classification of groups in which all noncyclic proper subgroups have the same order. For such groups, we have:

***Theorem* 3** — *Let G be a group having at least one noncyclic proper subgroups. Suppose that all noncyclic proper subgroups of G have the same order, then one of the following statements holds:*

(1) *G is one of groups in [1, Theorem 3.1];*

(2) $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$*, where $p \geq 2$ is a prime;*

(3) $G = D_8$*, the dihedral group of order* 8*;*

(4) $G = \langle a, b \mid a^p = b^p = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1\rangle$*, where p is an odd prime;*

(5) $G = Q_{16}$*, the quaternion group of order* 16*.*