Mathematics for the Life Sciences (MATH1134) - Tutorial Sheet 2

`This tutorial complements the material covered in Lecture 2.`

1. Solve as many of the following ODEs/IVPs as you need to be comfortable with the process. More examples and exercises are available in standard introductory differential equations books, such as `Elementary Differential Equations` by Boyce and Diprima *(Wiley)*:

(a) $t\dfrac{dy}{dt} - 3y = t^2 + t^4, \quad y(1) = 5$

(b) $x\dfrac{dx}{dt} - \sin(x)\dfrac{dx}{dt} = 2e^t, \quad x(0) = 0$

(c) $x\dfrac{dy}{dx} - 3y = 0, \quad y(1) = 8 \quad (x \geq 1)$

(d) $x\dfrac{dy}{dx} = 6x - 3xy, \quad y(1) = 1 \quad (x \geq 1)$

(e) $\dfrac{x}{y}\dfrac{dy}{dx} = 1 + y$

(f) $t\dfrac{dy}{dt} + (t+1)y = t$

(g) $\dfrac{dx}{dt}e^{3x} = 12e^{2x}t^2 - 16e^{2x}$

(h) $e^t\left(8 - \dfrac{dy}{dt}\right) = 17$

(a) By dividing both sides by t, this can be re-written as a *linear first order ODE* in standard form: $\boldsymbol{\dfrac{dy}{dt} - \dfrac{3}{t}y = t + t^3}$. The integrating factor is $e^{\int \frac{-3}{t}\,dt} = e^{-3\ln t}$ $\boldsymbol{= t^{-3}}$.

So $t^{-3}y(t) = \displaystyle\int t^{-3}(t + t^3)\,dt = \int (t^{-2} + 1)\,dt = -t^{-1} + t + C \Rightarrow y(t) = t^3\left[-t^{-1} + t + C\right] \Rightarrow$

$\boxed{\boldsymbol{y(t) = -t^2 + t^4 + Ct^3}}$

(b) By factoring on the left hand side, this can be written as $\boldsymbol{(x - \sin(x))\frac{dx}{dt} = 2e^t}$.

So this is a *separable ODE* and we have $\boldsymbol{\displaystyle\int (x - \sin(x))\,dx = \int 2e^t\,dt}$.

Thus $\boldsymbol{\dfrac{x^2}{2} + \cos(x) = 2e^t + C}$.

The initial condition means that $0 + 1 = 2 + C \Rightarrow$ $\boxed{\boldsymbol{C = -1}}$

(c) By dividing both sides by x, this can be re-written as a *linear first order ODE* in standard form: $\boldsymbol{\dfrac{dy}{dx} - \dfrac{3}{x}y = 0}$. The integrating factor is $e^{\int p(x)\,dx} = e^{-\int 3/x\,dx} =$ $e^{-\ln x^3} = \boxed{x^{-3}}$.

So $x^{-3}y(x) = \displaystyle\int 0\,dt = C \Rightarrow \boxed{y(x) = Cx^3}$ Finally, to find C we use $y(1) = 8 \Rightarrow$ $\boxed{C = 8}$.

ALTERNATIVELY: ______________________________

This could also be viewed as a separable equation: $\dfrac{1}{y}\,dy = \dfrac{3}{x}\,dx$

Integrate both sides to get $\ln|y| = 3\ln|x| + C_1 \Rightarrow y = Cx^3$

Finally, to find C we use $y(1) = 8 \Rightarrow \boxed{C = 8}$

(d) This is a linear ODE which in standard form is $\dfrac{dy}{dx} + 3y = 6$ **The integrating factor is** $e^{\int p(x)\,dx} = e^{\int 3\,dx} = \boxed{e^{3x}}$

Thus $e^{3x}y(x) = \displaystyle\int 6e^{3x}\,dx = 2e^{3x} + C \Rightarrow \quad y(x) = \quad e^{-3x}\left[2e^{3x} + C\right] \quad = \quad e^{-3x}\left\{2e^{3x}\right\} + Ce^{-3x}$

so $\boxed{\boldsymbol{y(x) = 2 + Ce^{-3x}.}}$.

The initial condition means that $1 = 2 + Ce^{-3}$ **therefore** $C = -e^3$ **and**

$$y(x) \quad = \quad 2 - e^3e^{-3x} \quad = \quad 2 - e^{3(1-x)}.$$

(e) $y = \frac{x}{C-x}$**.**

(f) $y = 1 - \frac{1}{t} + \frac{c}{t}e^{-t}$**.**

(g) By factoring, we have

$$\frac{dx}{dt}e^{3x} = e^{2x}(12t^2 - 16)$$

$$\Rightarrow \frac{dx}{dt}e^{x} = 12t^2 - 16 \qquad \text{or} \qquad dx\ e^x = (12t^2 - 16)\ dt.$$

So

$$\int e^x\ dx = \int (12t^2 - 16)\ dt \qquad \Rightarrow \qquad e^x = 4t^3 - 16t + C.$$

So by the laws of exponents

$$x(t) = \ln\left(4t^3 - 16t + C\right).$$

(h) This is an "integration in disguise" problem. The equation can easily be rearranged to

$$8 - \frac{dy}{dt} = 17e^{-t} \qquad \Rightarrow \qquad \frac{dy}{dt} = 8 - 17e^{-t} \qquad \Rightarrow$$

$$\int \frac{dy}{dt}\ dt = \int (8 - 17e^{-t})\ dt \qquad \Rightarrow \qquad y(t) = 8t + 17e^{-t} + C.$$

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2. Find all equilibrium solutions of

$$\frac{dy}{dt} = (y+1)^2(y^2 - 1)$$

and use calculus to classify each equilibrium solution as stable, unstable, or semistable.

Solving $f(y) = (y+1)^2(y^2-1) = 0$, we get two equilibrium solutions $\boxed{\boldsymbol{y = -1, 1}}$.

$$f'(y) = 2(y+1)(y^2-1) + 2y(y+1)^2$$

so

$$f'(1) = 2(2)(0) + 2(1)(2^2) = 8 > 0 \quad \Rightarrow \quad \mathit{UNSTABLE\ EQUILIBRIUM}.$$

$$f'(-1) = 2(0)(0) + 2(-1)(0) = 0 \quad \Rightarrow$$

we need additional information to classify this equilibrium point. We observe that for $y \in (-\infty, -1)$ $f(y) = \frac{dy}{dt} > 0$ and for $y \in (-1, 1)$, $f(y) = \frac{dy}{dt} < 0$ so that all solutions $y(t)$ which start out *below* the equilibrium solution $y = -1$ are increasing functions of t, whereas all solutions $y(t)$ which start out *above* the equilibrium solution $y = -1$ are decreasing functions of t. Hence $y = -1$ is a *STABLE EQUILIBRIUM*.

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3. For the initial value problem

$$\frac{dy}{dt} = 10y^2(y^2 - 1), \qquad y(0) = y_0 \quad (where\ y_0 \in \mathbb{R}),$$

determine the value of all equilibrium points and state, *with reason*, whether each equilibrium point is stable, unstable, or semistable.

Solving $f(y) = 10y^2(y^2 - 1) = 0$, we get three equilibrium points $\boxed{\boldsymbol{y = -1, 0, 1}}$.

$$f'(y) = 20y(y^2 - 1) + 20y^3$$

so

$$f'(-1) = 0 + 20(-1)^3 = -20 \quad \Rightarrow \quad STABLE\ EQUILIBRIUM.$$
$$f'(1) = 0 + 20(1)^3 = 20 \quad \Rightarrow \quad UNSTABLE\ EQUILIBRIUM.$$

and since $f'(0) = 0$, so we need additional information to classify this equilibrium point. We observe that for $y \in (-1, 0)$ and $y \in (0, 1)$, $f(y) = \frac{dy}{dt} < 0$ so that y is a decreasing function of t for all solutions y on both sides of the equilibrium. Hence $y = 0$ is an *SEMISTABLE EQUILIBRIUM*.

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4. This question requires the plotting of *direction fields* for first order ODEs of the form $\frac{dy}{dt} = f(t, y)$, **which you can do using the website given in the class notes**.

However, for us it is preferable to plot the *direction fields* using MATLAB. Do the following:

(a) Go to the class Moodle page and download **dirfield_arrow_funchandle.m** and **dirfield_funchandle.m** from the *Matlab Files* folder in the section containing *Lecture 2*. (These files are slight modifications of **dirfield.m** from the website http://terpconnect.umd.edu/~petersd/246/dirfield.m, which work with **function handles** instead of the soon-to-be-obsolete-from-MATLAB *inline functions*).

(b) Look at the (small) files **dirfield_arrow_funchandle.m** and **dirfield_funchandle.m**; they are function M files which makes use of the MATLAB function `quiver()` to generate the direction fields.

(c) To use **dirfield_arrow_funchandle.m** or **dirfield_funchandle.m**, you will first have to write a function handle for the right-hand-side function of the differential equation, $\frac{dy}{dt} = f(t, y)$. You can do this on the command line or in a script M-file which later calls the function **dirfield_arrow_funchandle** or **dirfield_funchandle**. NOTE that the function handles used in **dirfield_arrow_funchandle.m** and **dirfield_funchandle.m** MUST be vectorised (so use $.*$ instead of $*$ etc.). For example

```
f = @(t,y) -y .^ 2 + t./2
```

for $\frac{dy}{dt} = -y^2 + \frac{t}{2}$.

(d) You then need to define a vector of t values and a vector of y values which determine the rectangular grid on which you wish to see the direction field. For example

```
t = 0:0.2:3;
y = -3:0.4:3;
```

would cause the direction field to be plotted in the rectangle $0 \leq t \leq 3$ and $-3 \leq y \leq 3$ on the 16×16 grid determined by $\Delta t = 0.2$ and $\Delta y = 0.4$.

(e) Finally, the function **dirfield_arrow_funchandle()** or **dirfield_funchandle()** should be called as follows:

```
dirfield_arrow_funchandle(f, t, y)
```

or

```
dirfield_funchandle(f, t, y)
```

The first argument should be the name of the function handle, the second argument the vector of t values, and the third argument the vector of y values.

(f) You should also read the comments at the top of the files **dirfield_arrow_funchandle.m** and **dirfield_arrow_funchandle.m** for more about how to use them.

In the following, plot the requested direction field on the suggested rectangle and write down any *equilibrium solutions* you can identify, also stating whether they are stable, unstable, or semistable.

(i) $\dfrac{dy}{dt} = 2y - 30$ on the rectangle $0 \leq t \leq 10$, $0 \leq y \leq 30$.

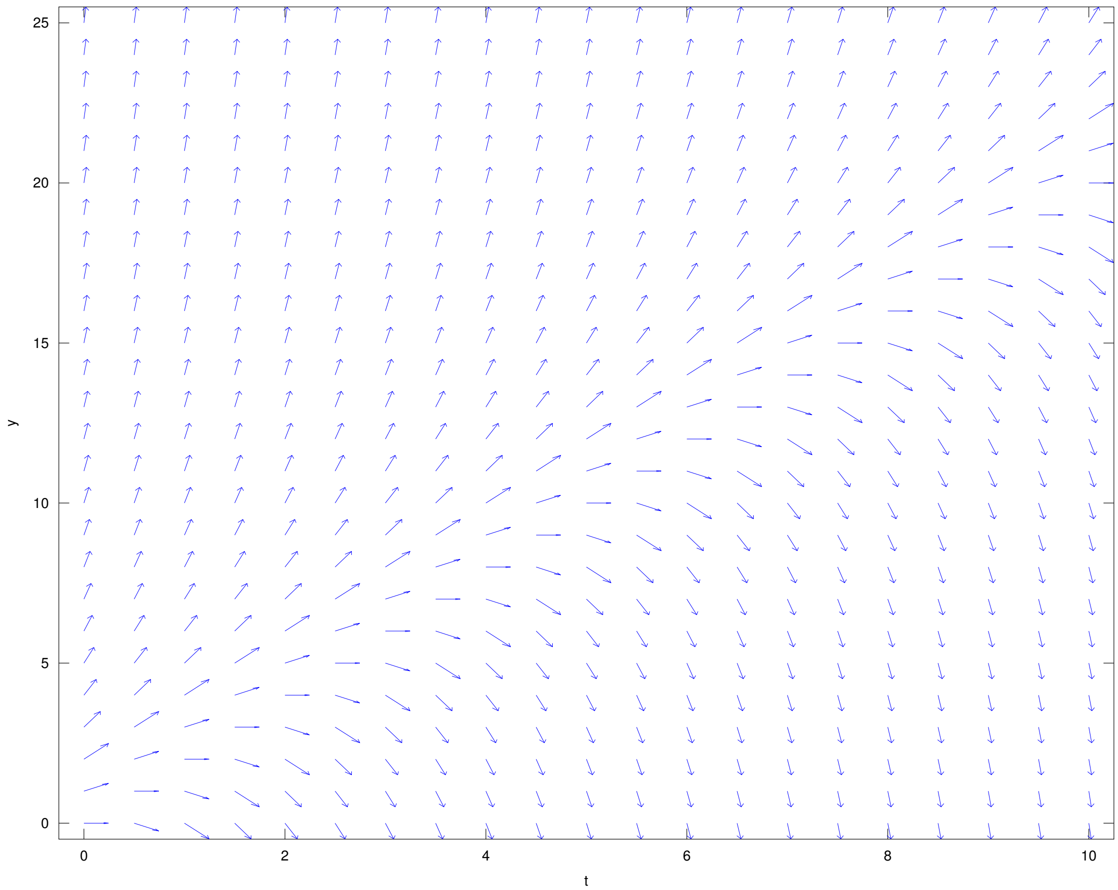
(ii) $\dfrac{dy}{dt} = y^2(y+4)$ on the rectangle $0 \leq t \leq 10$, $-5 \leq y \leq 5$.

(iii) $\dfrac{dy}{dt} = y - 2t$ on the rectangle $0 \leq t \leq 10$, $0 \leq y \leq 25$.

(i) You should see an unstable equilibrium solution at $y = 15$.

(ii) You should see a semistable equilibrium solution at $y = 0$ and an unstable equilibrium solution at $y = -4$.

(iii) You should see something like this



5. Plot a direction field for the ODE from Question 2, $\frac{dy}{dt} = (y+1)^2(y^2-1)$, on an appropriate rectangle, and thus verify the existence and classification of the equilibrium solutions obtained in your answers to Question 2.

6. Write a MATLAB program which implements Euler's method, Heun's method, and the fourth order Runge-Kutta (RK4) method in the same M-file. Your program should be written to solve the generic first order IVP

$$\frac{dy}{dt} = f(t,y), \quad t \in [t_0, T], \quad y(t_0) = y_0$$

where values such as t_0, T, the number of steps N or step-size $h = (T - t_0)/N$ are either set in the program or input by the user. It is recommended that you write $f(t,y)$ as a function handle at the top of your M-file so that it is self contained *(ask me if you want to know how you can write your program so that the function handle is input by the user)*. Give your program the following features:

(a) The user selects whether to do all three methods at once or just one of the methods, or, since we will mainly use Heun's method and RK4, whether to only do those two methods.

As an example, the user could be asked to input 1 to do Euler's method only, 2 to do Heun's method only, 3 to do the RK4 method only, 4 to do all three methods, or 5 to do just Heun's and RK4.

(b) At a minimum, for each approximation method the output should include a table of the time step number, the time, the approximation Y_i at that time, the exact value of the solution function at that time *(just put in a dummy function and ignore this and the next column if the exact solution is not known)*, and the error at that time (like the examples done in class).

(c) Also produce a plot of the the approximate and exact solution (if known) on the same axes.

Try your program on the example done in the lecture (EXAMPLE 10) as well as some of the IVPs from Question 1.

NOTE shortly after Lecture 2 you will be provided with a program which does what is requested above but it is good practice to try to write one yourself first.

In particular, see the *Matlab Files* folder under *Lecture 2* on the class Moodle page where `EulerHeunRK4.m` is a script M file which does what is requested in this question.

For those students who want a more self-contained program (similar to `dirfield_arrow_funchandle`, discussed in Question 4 of this tutorial), I also include `EulerHeunRK4_function.m` which is a function M file. Type `EulerHeunRK4_function` and then hit `Enter` in the command window to see instructions on how to use that function. Its use is similar to in-built Matlab functions like `ode45` - which are discussed in *Tutorial 3.* A typical use of `EulerHeunRK4_function` would be

```
>> Y = EulerHeunRK4_function(f, t, y0, g)
```

where f is a function handle for the right side of the ODE $\frac{dy}{dt} = f(t, y)$, t is the vector of t values at which y is to be approximated, $y0$ is the initial value of the solution function $y(t)$, g is a function handle for the true solution function $y(t)$, and Y would be either a vector or matrix in which the approximate solution(s) is/are stored.

Alternatively, if one does not know the true solution and only wants the approximations, then

```
>> Y = EulerHeunRK4_function(f, t, y0)
```

will also work.