TIPS

1. Invest time learning the *language* of Mathematics (with all of its special cases and exceptions and conventions).
2. Review constantly.
3. Do assignments, tutorials, etc. Practise, practise, practise. Read textbook/supplementary notes. Ideally read the textbook/supplementary notes material on a topic *before* the relevant lecture. You will have to do significant work outside of the classroom to master the material.
4. Attend and engage with lectures and tutorials. This represents the most efficient way to learn the material. Come prepared and ask questions if you do not understand something.
5. Speak to me as soon as you feel you may be falling behind.
6. In summary: KEEP UP. This will be a very fast-paced class and falling behind is **very unwise**. Mathematics is very hierarchical and you generally progress only by first knowing well what went before.

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  1. A printable version with parts missing. *You are expected to print this out and read it BEFORE the lecture and to try to do the examples.* **Bring this printout with you to the lecture in order to complete the notes.**
  2. A complete printable version will be made available (some time) after lectures.
  3. A complete in-class version which attempts to mimic the way the lecture was delivered in class. I recommend using this version if you miss a lecture to get caught up on that lecture.

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$\rightsquigarrow$ *This simple introduction will give you the tools to explore other applications of mathematics to life sciences, such as biochemical (including enzyme) kinetics, and I will point some of these out to you as we proceed:*

Suggested Reading List

- **Mathematics of Life: Unlocking the Secrets of Existence** by Ian Stewart *(Profile)*.
- **Essential Mathematical Biology** by Nicholas Britton *(Springer)*
- **Mathematical Biology: I. An Introduction** by J.D. Murray *(Springer)*
- **Mathematical Models in Biology** by Leah Edelstein-Keshet *(SIAM)*
- **A Primer on Mathematical Models in Biology** by Lee Segel and Leah Edelstein-Keshet *(SIAM)*
- **Super Cooperators: Evolution, Altruism and Human Behaviour or Why We Need Each Other to Succeed** by Martin Nowak and Roger Highfield *(Canongate)*
- **When maths doesn't work: what we learn from the Prisoners' Dilemma** by Tony Mann - *Lecture transcript and video available at http://www.gresham.ac.uk/lectures-andevents/when-mathsdoesnt-work-what-welearn-from-theprisoners-dilemma;*

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- **As always, NOTATION and LANGUAGE will be very important in what follows; pay close attention to it and ensure that you learn and understand the notation and language used!**

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We will also study systems of ODEs in this course.

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$\mapsto$ **It is important that you learn how to recognise which of the three categories a given differential equation falls into so that you will know exactly how to solve it** *(or whether you can solve it)***!**

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  ⇝ For example $\frac{dy}{dx} = f(x) \Rightarrow \int \frac{dy}{dx}\,dx = \int f(x)\,dx$ OR $\boldsymbol{y = \int f(x)\,dx}$.

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CHECK YOUR ANSWER!!!

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Substitute this into the general solution to get $3 = 6 - \frac{5}{2} + C \Rightarrow C = -\frac{1}{2}$ and the particular solution is $z(y) = 10y + 6e^{-y} - \frac{5}{2}e^{-2y} - \frac{1}{2}$.

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The acceleration of a bus along a straight road is given by $a(t) = 3t$. We start observing the bus ($t = 0$) when it is 50 m along the road, and 1 s after we begin observing it, its velocity is 5 m/s. Write this as an ODE with "initial" conditions, and find a particular solution for the *position* of the bus along the road at time t.

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$$x(t) = \frac{1}{2}t^3 + \frac{7}{2}t + 50.$$

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- ↪ NOTE after integration, only **ONE** constant of integration is needed in Equation (2), and it is not always feasible to solve the resulting equation for y as function of x.

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$$\int \frac{1}{g(y)} dy = \int \frac{1}{g(y(x))} \frac{dy}{dx} dx$$
$$= \int \frac{1}{g(y(x))} f(x) g(y(x)) dx = \int f(x) dx.$$

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$\hookrightarrow$ For those who are worried that we are integrating with respect to *different* variables on both sides of the equation, here is why it works *(from the substitution rule . . . this proof is also available in most calculus textbooks)*:

$$\int \frac{1}{g(y)} dy = \int \frac{1}{g(y(x))} \frac{dy}{dx} dx$$

$$= \int \frac{1}{g(y(x))} f(x) g(y(x)) dx = \int f(x) dx.$$

- **NOTE** this method of solution is sometimes referred to as **separation of variables**.

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So, returning to Equation 3, we have $\frac{du}{dt} = e^u e^{3t} = e^{u+3t}$, so that our solution *does* satisfy the DE.

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- Note the difficulty of finding an explicit solution in this case.

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- The modified linear first order ODE would then look like

$$\mathrm{I}(t)\frac{dy}{dt} + \mathrm{I}(t)p(t)y = \frac{d}{dt}\left[\mathrm{I}(t)y\right] = I(t)r(t) \tag{6}$$

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Ignoring constants of integration (we only need *one* function $\mathrm{I}(t)$ to serve as the integrating factor), we solve to get $\boldsymbol{\ln\left[\mathrm{I}(t)\right] = \int p(t)\,dt \Rightarrow \mathrm{I}(t) = e^{\int p(t)\,dt}}$.

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- Hence, the integrating factor to choose when solving $\frac{dy}{dt} + p(t)y = r(t)$ is **always**

$$\mathrm{I}(t) = e^{\int p(t)\, dt}. \qquad (7)$$

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- The ODE, Equation (6), then becomes $\frac{d}{dt}\left(e^{\int p(t)\,dt}y\right) = e^{\int p(t)\,dt}r(t)$, and by the Fundamental Theorem of Calculus we integrate both sides with respect to t to get
$$e^{\int p(t)\,dt}y = \int \left[e^{\int p(t)\,dt}r(xt\right]\,dx + C, \tag{8}$$
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$\hookrightarrow$ Instead of focussing on the above formula, I recommend remembering the *process* of multiplying the equation by an integrating factor to get the product rule in reverse, then using the **Fundamental Theorem of Calculus** to solve the resulting modified ODE.

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- Can you think of a first order ODE which could be considered both *linear* and *separable*?

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  And as in EXAMPLE 7, the initial condition means $C = 1$.

- Can you think of a first order ODE which could be considered both *linear* and *separable*? See **Tutorial 2** for some more cases of ODEs which fall into more than one of the three categories we have considered.

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- For example, the separable ODEs in EXAMPLE 4 ($\frac{du}{dt} = e^{u+3t}$) and EXAMPLE 5 ($y' = \frac{y\cos x}{1+2y^2}$) are *nonlinear* since they cannot be written in the form $\frac{dy}{dt} + p(t)y = r(t)$.

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$\hookrightarrow$ If f and $\dfrac{\partial f}{\partial y}$ are *continuous* in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0), then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there exists a unique solution to the above IVP.

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$\rightsquigarrow$ A proof of this is quite advanced and is omitted. A simplified version can be found in section 2.8 of `Elementary Differential Equations` by *Boyce and Diprima* and more thorough proofs and discussions of existence and uniqueness of solutions can be found in books such as `Ordinary Differential Equations` by *Birkhoff and Rota* (for example, in sections 10-12 of chapter 1).

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- In the case of linear first order IVPs $y' + p(t)y = r(t)$, $\;y(t_0) = y_0$, the above theorem becomes: If p and g are continuous on an open interval containing the point $t = t_0$, then there exists a unique solution to the IVP on that same open interval containing t_0.

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$\mapsto$ **Introduction**

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- This is often the case, for example, with population models where we sometimes just want to see the long term trend - *i.e.*, what happens to the population as time $\to \infty$, or see if there are certain threshold population values on either side of which the population evolution is very different.
- **KEY DEFINITION**: If a *first order* ODE can be written in the form $\boxed{\frac{dy}{dt} = f(y),}$ so that the right-hand-side is a function of the dependent variable y only, then the ODE is called **autonomous**. Otherwise, it is called non-autonomous (or just *not autonomous*).

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$\mapsto$ In an autonomous ODE $\dfrac{dy}{dt} = f(y)$, if $f(y)$ is differentiable *(hence continuous)* then any **root(s)** of $f(y)$ is/are called equilibrium point(s) or critical point(s) of the ODE.

↦ If $y = y_0$ is an equilibrium point of $\frac{dy}{dt} = f(y)$, so

$$\boldsymbol{\frac{dy}{dt} = f(y_0) = 0}$$

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↦ **KEY DEFINITION** For autonomous ODE

$$\frac{dy}{dt} = f(y),$$

the (constant) solutions to $f(y) = 0$ are called equilibrium points, critical points, or equilibrium solutions to the ODE.
Thus if $f(y_0) = 0$ then $y(t) = y_0$ is an equilibrium solution of the ODE $\frac{dy}{dt} = f(y)$.

- On a graph of solutions y versus t of the ODE $\frac{dy}{dt} = f(y)$ where we are only interested in solutions over the interval $t \geq t_0$, the graphs of the equilibrium solutions would be **horizontal lines** which would separate the plane (for $t \geq t_0$) into regions where the general behaviour as $t \to \infty$ of other solutions would be the same.

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Example:

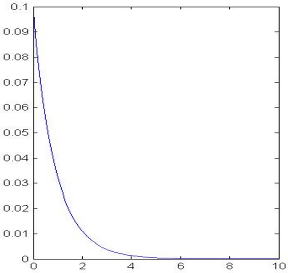
$$\frac{dy}{dt} = \frac{y}{y-1}, \qquad y \neq 1$$

clearly $y_0 = 0$ is an equilibrium point. It is not difficult to solve this equation and the solution is

$$y(t) - \ln y(t) = t + C$$

but we cannot write $y(t)$ as an explicit function of t . Despite this we know *from the equation* that there is a constant zero solution for a zero starting value.

What happens for this case if we start off with a **nearby** non-zero value e.g. 0.1 or -0.1 ? The solution simply moves to the equilibrium value i.e. this value is in some sense stable.



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  - **semistable** if some solutions $y(t)$ which start out near to y_0 move closer to y_0 and some move away from y_0 as $t \to \infty$. In general, solutions that start out below y_0 will move away from y_0 AND solutions that start out above y_0 will move towards y_0 OR vice versa.

The basic rules for classifying an equilibrium solution y_0 of $\frac{dy}{dt} = f(y)$ as *stable*, *unstable*, or *semistable* can be easily figured out using basic calculus and the aid of two related graphs to figure out how solutions starting out close to that equilibrium point behave:

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$\mapsto$ In the upcoming few slides, the images are, without loss of generality, for an equilibrium point of $y_0 = 2$.

- The key to understanding the behaviour of solutions close to a critical/stationary/equilibrium point, $y = y_0$, is *to look at the gradient of the RHS at that equilibrium point - i.e.*

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- Understanding the points above is key to easily classifying the *equilibrium points* of an autonomous ODE as **stable**, **unstable**, or **semi-stable**, so make a special effort to understand the geometry and calculus behind the next few slides.

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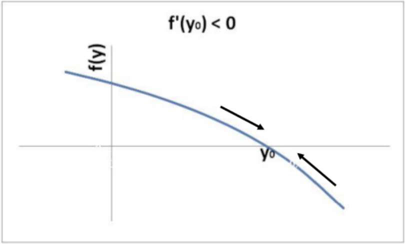

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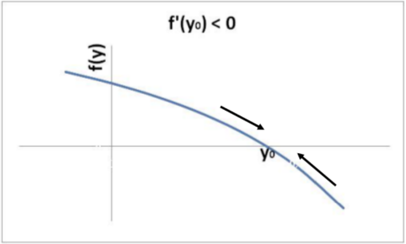

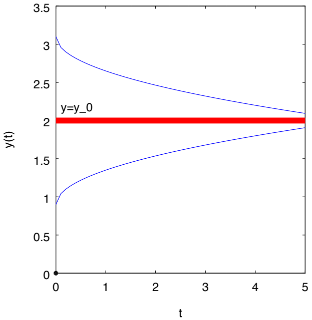

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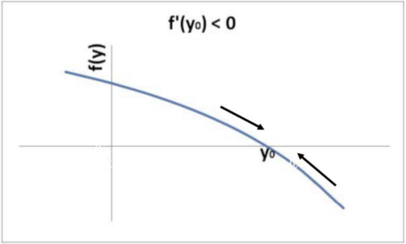

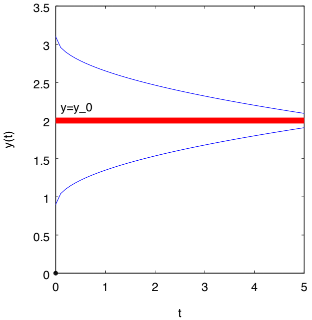

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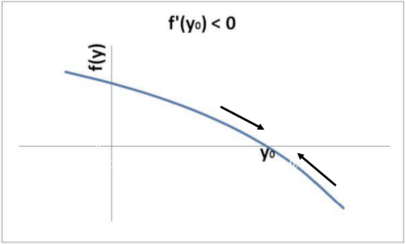

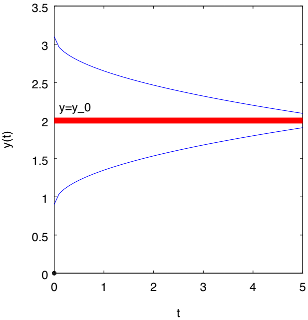

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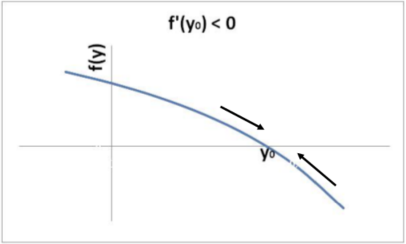
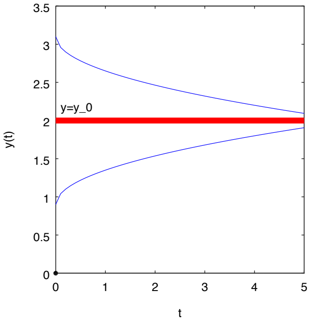
$\hookrightarrow$ for solutions $y(t) < y_0$ we have $f(y) = \frac{dy}{dt} > 0$ so such solutions are *increasing* towards y_0 as $t \to \infty$;

$\boxed{f'(y_0) < 0}$





$f(y)$ is *decreasing* through 0 at $y = y_0$, therefore

- $\hookrightarrow$ for solutions $y(t) < y_0$ we have $f(y) = \frac{dy}{dt} > 0$ so such solutions are *increasing* towards y_0 as $t \to \infty$;
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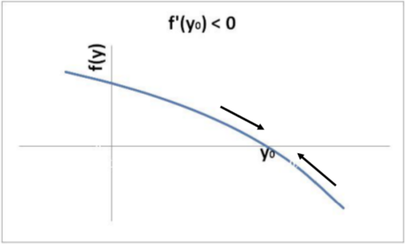
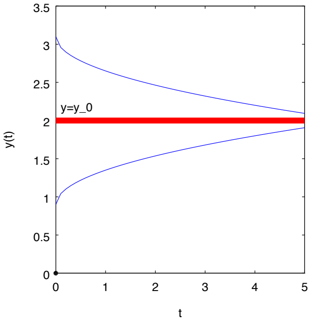
$$\boxed{f'(y_0) < 0}$$





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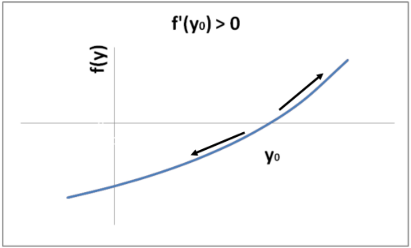
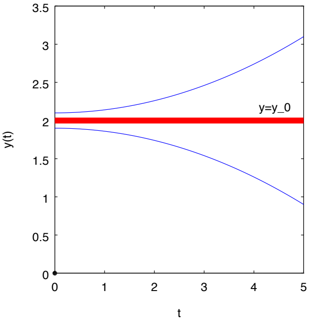
- In summary, **solutions starting out near to** y_0 **move towards** y_0, $\cdots$

HENCE

$$\boxed{\begin{array}{c} f'(y_0) < 0 \\ \Downarrow \\ y_0 \text{ is a STABLE equilibrium point} \end{array}}$$

$$\boxed{f'(y_0) > 0}$$





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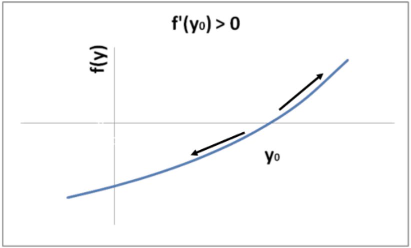

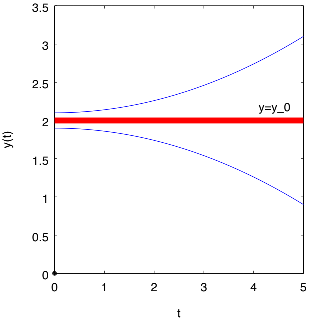

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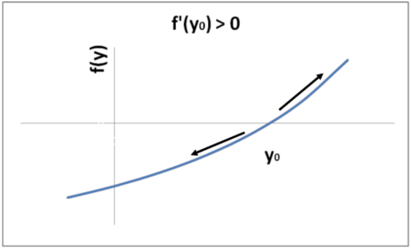

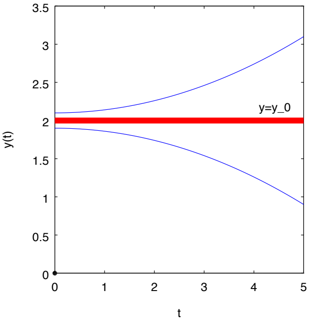

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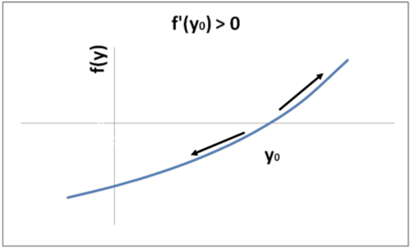

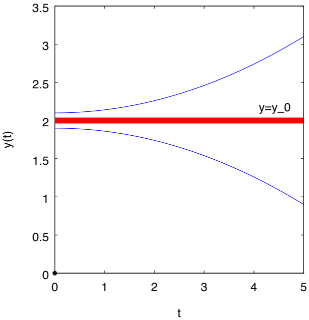

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HENCE

$$\boxed{\begin{gathered} f'(y_0) > 0 \\ \Downarrow \\ y_0 \text{ is an } \textbf{UNSTABLE} \text{ equilibrium point} \end{gathered}}$$

$$\boxed{f'(y_0) = 0}$$

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$\hookrightarrow$ This is an ambiguous case and requires a careful inspection of the ODE - in particular, of the **SIGN** **(+ve or −ve) of** $\boxed{f(y) = \frac{dy}{dt}}$ **immediately to the left and right of** $y = y_0$.

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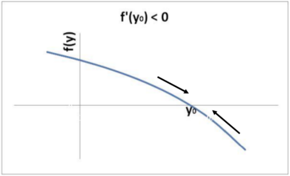

1. If $f(y)$ is positive to the left of y_0 and negative to the right of y_0, then $\boxed{y_0 \text{ is a } \textbf{STABLE} \text{ equilibrium point}}$.

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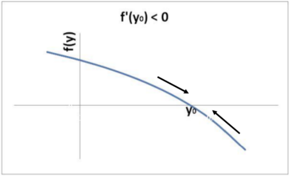

1. If $f(y)$ is positive to the left of y_0 and negative to the right of y_0, then $\boxed{y_0 \text{ is a } \textbf{STABLE} \text{ equilibrium point}}$.

- E.g. $y_0 = 0$ and $f(y) = -2y^5$.

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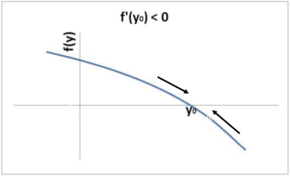

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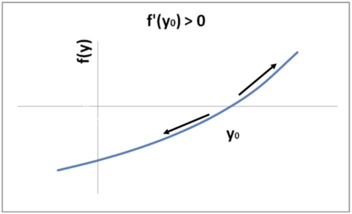

2. If $f(y)$ is negative to the left of y_0 and positive to the right of y_0, then $\boxed{y_0 \text{ is an } \textbf{UNSTABLE} \text{ equilibrium point}}$.

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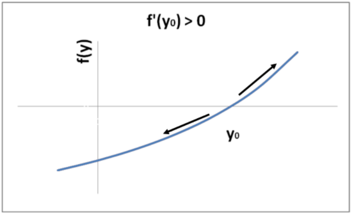

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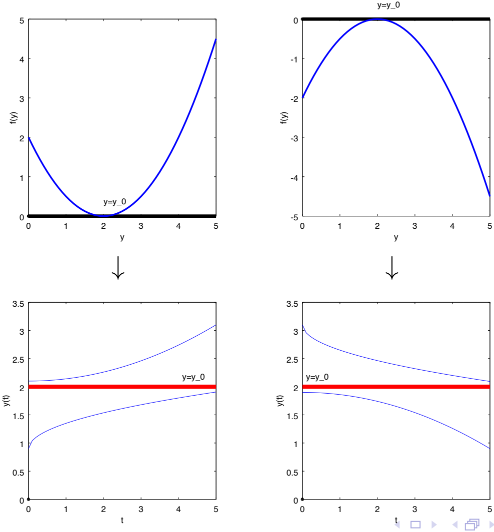

3. **NEW CASE - SEMI-STABLE EQUILIBRIUM POINT** **If** $f'(y_0) = 0$ **and** $f(y)$ **is negative to the left AND right of** y_0 **OR** $f(y)$ **is positive to the left AND right of** y_0**, then** $\boxed{y_0 \text{ is an SEMI-STABLE equilibrium point}}$.

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In this case, starting values of y on *one side* of the equilibrium point, y_0, will approach y_0 as $t \to \infty$ **(stable)**, and starting values of y on the *other side* of y_0 will move away from y_0 as $t \to \infty$ **(unstable)**.

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- E.g. $\boldsymbol{y_0 = 4}$ and $\boldsymbol{f(y) = (y-4)^2}$. Here, $\boldsymbol{f(y) = \frac{dy}{dt} > 0}$ on *both* sides of the equilibrium point, hence $\boldsymbol{y(t)}$ is **increasing** on both sides of the equilibrium point. So for $\boldsymbol{y < y_0}$, the solutions $\boldsymbol{y(t)}$ approach the equilibrium point **(stable)**, and for $\boldsymbol{y > y_0}$ the solutions $\boldsymbol{y(t)}$ move away from the equilibrium point **unstable**, hence $\boldsymbol{y_0}$ is an **SEMI-STABLE** equilibrium point.

- $\boxed{\text{EXAMPLE 8}}$ *(From a MATH1106 Tutorial)* Find all equilibrium points of the following differential equations, and use calculus to classify each equilibrium point as stable, unstable, or semi-stable:

(a) $\dfrac{dy}{dt} = (y^2 - 4)(y^2 - 25)(y + 2)$.

(b) $\dfrac{dy}{dt} = e^{3y}$.

(c) $\dfrac{dy}{dt} = e^{3y} - e$.

(d) $\dfrac{dy}{dt} = (y - 4)^3 \ln(y^2 + 1)$.

(a) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get

$$(y^2 - 4)(y^2 - 25)(y + 2) = 0 \quad \text{or} \quad (y - 2)(y + 2)^2(y - 5)(y + 5) = 0$$

so that the equilibrium points are $y = -5, -2, 2, 5$.

(a) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get

$$(y^2 - 4)(y^2 - 25)(y + 2) = 0 \quad \text{or} \quad (y - 2)(y + 2)^2(y - 5)(y + 5) = 0$$

so that the equilibrium points are $y = -5, -2, 2, 5$.
Next, by the product rule,

$$\frac{df}{dy} = f'(y) = 2y(y^2 - 25)(y + 2) + 2y(y^2 - 4)(y + 2) + (y^2 - 4)(y^2 - 25).$$

(a) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get

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$$f'(-5) = 630 > 0 \quad \Rightarrow$$

(a) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get

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$$f'(-5) = 630 > 0 \quad \Rightarrow \quad -5 \text{ is an UNSTABLE equilibrium point.}$$

$$f'(2) = -336 < 0 \quad \Rightarrow$$

(a) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get

$$(y^2-4)(y^2-25)(y+2) = 0 \quad \text{or} \quad (y-2)(y+2)^2(y-5)(y+5) = 0$$

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$$f'(-5) = 630 > 0 \quad \Rightarrow \quad -5 \text{ is an UNSTABLE equilibrium point.}$$

$$f'(2) = -336 < 0 \quad \Rightarrow \quad 2 \text{ is a STABLE equilibrium point.}$$

(a) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get

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$$f'(2) = -336 < 0 \quad \Rightarrow \quad 2 \text{ is a STABLE equilibrium point.}$$

$$f'(5) = 1470 > 0 \quad \Rightarrow \quad 5 \text{ is an UNSTABLE equilibrium point.}$$

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$$f'(-5) = 630 > 0 \quad \Rightarrow \quad -5 \text{ is an UNSTABLE equilibrium point.}$$

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$$\text{Meanwhile } f'(-2) = 0 \quad \Rightarrow \quad \text{we have to investigate further.}$$

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We pick a point to the left and a point to the right of the equilibrium point, $y = -2$, and see what the *sign* of $\frac{dy}{dt} = f(y) = (y^2-4)(y^2-25)(y+2)$ is.

(a) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get

$$(y^2 - 4)(y^2 - 25)(y + 2) = 0 \quad \text{or} \quad (y - 2)(y + 2)^2(y - 5)(y + 5) = 0$$

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We pick a point to the left and a point to the right of the equilibrium point, $y = -2$, and see what the *sign* of $\frac{dy}{dt} = f(y) = (y^2 - 4)(y^2 - 25)(y + 2)$ is. For example, for $y = -3$ we get $f(-3) = 80 > 0$ AND for $y = 0$ we get $f(0) = 200 > 0$ so that $\frac{dy}{dt} = f(y)$ *is the same sign* on both sides of the equilibrium point $y = -2$, hence $\boldsymbol{y = -2}$ *is a SEMISTABLE equilibrium point.*

(b) Answer: e^{3y} is never zero so there is NO equilibrium solution to this differential equation.

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(c) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get $e^{3y} = e \Rightarrow$, taking natural logarithms of both sides of the equation,

$$3y = 1 \Rightarrow y = \frac{1}{3}.$$

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$$3y = 1 \Rightarrow y = \frac{1}{3}.$$

Next, $\frac{df}{dy} = 3e^{3y} \Rightarrow f'\left(\frac{1}{3}\right) = 3e > 0$, therefore $y = \frac{1}{3}$ is an UNSTABLE equilibrium point.

(d) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get $(y-4)^3 \ln(y^2+1) = 0$. One obvious solution is $y = 4$, but we could also have $\ln(y^2+1) = 0$ if $y^2 + 1 = 1 \Rightarrow y = 0$.
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Next, using the product and chain rules,

$$\frac{df}{dy} = f'(y) = 3(y-4)^2 \ln(y^2+1) + \frac{2y}{y^2+1}(y-4)^3.$$

So $f'(0) = 0$ and $f'(4) = 0$, and both equilibrium points have to be examined more closely.

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$$f(-1) = -125 \ln(2) < 0 \quad \text{AND} \quad f(1) = -27 \ln 2 < 0$$

so $y = 0$ is a SEMISTABLE equilibrium point.

(d) Answer: Setting $\frac{dy}{dt} = 0$ and solving for y we get $(y-4)^3 \ln(y^2+1) = 0$. One obvious solution is $y = 4$, but we could also have $\ln(y^2+1) = 0$ if $y^2 + 1 = 1 \Rightarrow y = 0$.
So the two equilibrium solutions are $y = 0$ and $y = 4$.
Next, using the product and chain rules,

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so $y = 0$ is an UNSTABLE equilibrium point.

Classifying Equilibrium Solutions via LINEARISATION *(Taylor series again!)*

$\hookrightarrow$ An alternative way to arrive at earlier conclusions concerning classifying an equilibrium solution, y_0, of $\dfrac{dy}{dt} = f(y)$ by looking at $f'(y_0)$, is to consider a **Taylor series** expansion about y_0 of a *nearby solution* $\boldsymbol{y(t) = y_0 + \eta(t)}$, where $\eta(t)$ represents a small perturbation of the equilibrium solution $y(t) = y_0$.

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- So, taking the last equation in the sequence and using Taylor's theorem, provided $\eta(t)$ is smooth enough we have that

$$\frac{d\eta}{dt} = f(y_0+\eta(t)) = f(y_0)+f'(y_0)\eta(t)+f''(y_0)\frac{\eta(t)^2}{2!}+\textit{Higher Order Terms}$$

$\hookrightarrow$ Given that $f(y_0) = 0$ and if $\eta(t)$ is small enough for us to ignore all nonlinear higher order terms and truncate the Taylor series after the linear term *(hence the word "linearisation")*, then we see that (approximately) $\eta(t)$ satisfies the linear and separable ODE

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- **NOTE** we also see that if $f'(y_0) = 0$ then the nonlinear terms in the Taylor series expansion determine the stability properties of equilibrium solution $y(t) = y_0$.

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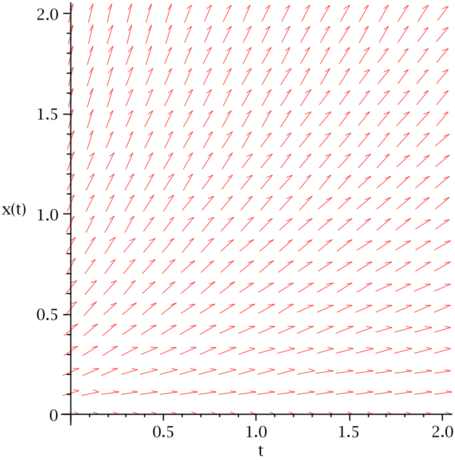
$t \rightarrow$ $x \downarrow$	**0**	**0.5**	**1**	**1.5**	**2**
0	0	0	0	0	0
0.5	1	2/3	1/2	2/5	1/3
1	2	4/3	1	4/5	2/3
1.5	3	2	3/2	6/5	1
2	4	8/3	2	8/5	4/3



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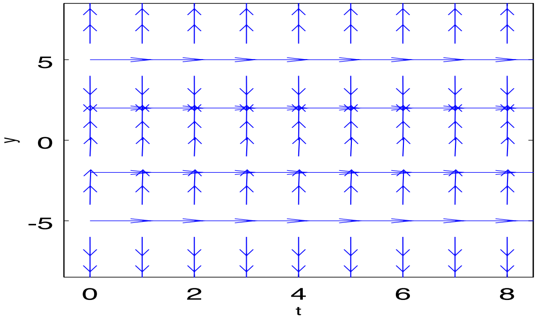

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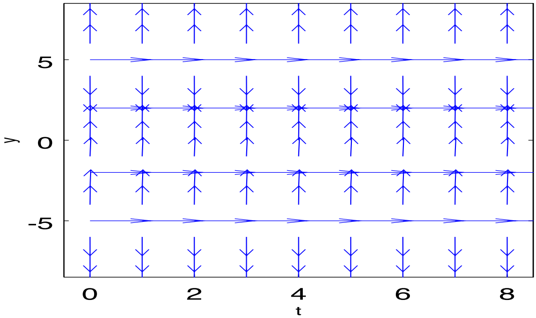

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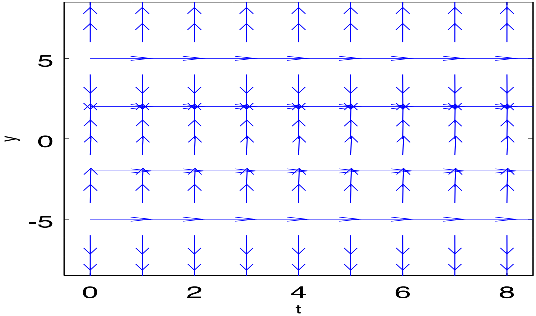

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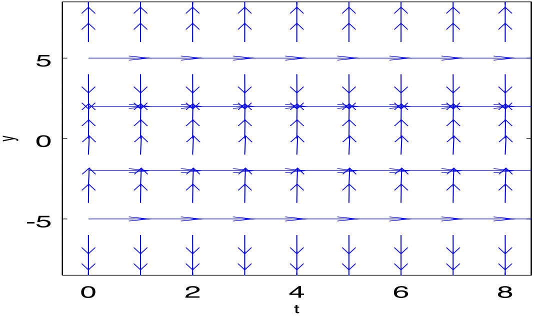

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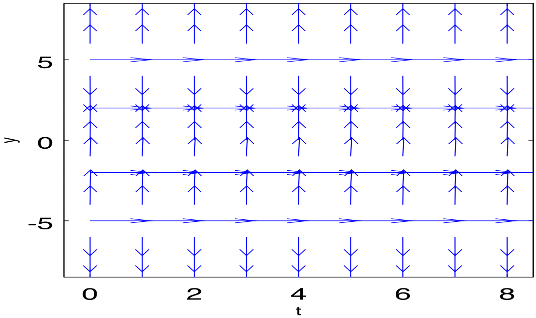

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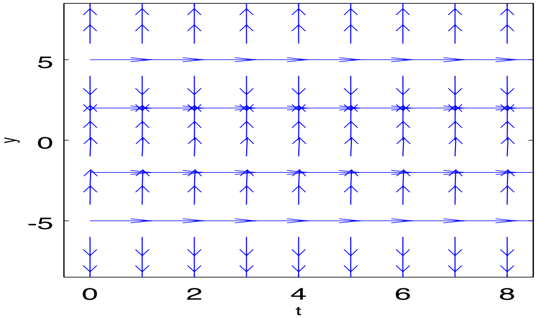

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$\hookrightarrow$ Optionally and preferably, you can also use MATLAB to draw direction fields, using the `quiver()` function.

- For more on this see **Tutorial 2**.

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- These numerical methods will be helpful as tools to investigate numerical solutions to IVPs and systems of IVPs which are difficult or impossible to solve analytically.

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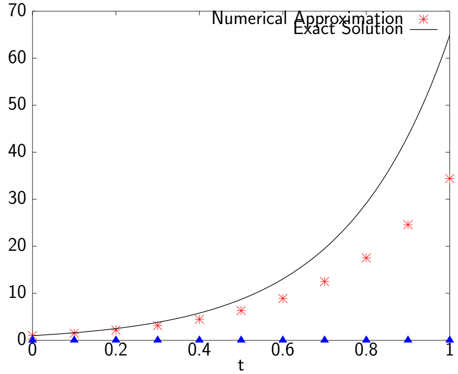
- **As always, NOTATION and LANGUAGE will be very important in what follows; pay close attention to it and ensure that you learn and understand the notation and language used!**



Here the unknown function $y(t)$ is approximated at only 11 points, $\boxed{t_0 = 0,\ t_1 = 0.1,\ t_2 = 0.2,\ \ldots,\ t_{10} = 1}$ *(indicated by the* ▲ *symbols on the* $t - axis$*)*, and the approximate value of the function at each t_i is indicated by the $*$ symbols.

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$\rightsquigarrow$ Since a numerical method gives only an *approximation* to the true solution function $y(t)$ at certain t values, we use a different notation to refer to this approximate solution:

$$Y_i \text{ or } y_i \quad \approx \quad y(t_i) \quad \text{for } i = 0, 1, 2, \ldots, N.$$

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  - *For example, a loop that runs from* 0 *to* N *in the notes would have to run from* 1 *to* $N+1$ *in MATLAB.*

- However, to make life easier, I will endeavour to always give the main formula for a numerical method in the standard form with the **0 to N indexing**, *THEN* in MATLAB form with **1 to N+1 indexing** (and vector notation).

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- See Tutorial 2 for programs corresponding to Euler's method, Heun's Method, and the fourth order Runge-Kutta method.

  - The method is derived from truncating a Taylor series expansion of the (unknown) solution function $y(t)$.

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$y_0 = y(t_0)$ THEN	$y_0 = y(t_0)$ THEN
	$\tilde{y}_{i+1} = y_i + hf(t_i, y_i)$ AND
$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))\right]$	$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})\right]$
for $i = 0, 1, 2 \ldots, N-1$.	for $i = 0, 1, 2 \ldots, N-1$

↪ In MATLAB syntax this would be: **Heun's Method (MATLAB version)**

ONE-STEP	TWO-STEP
$t = t_0 : h : T$, AND $y(1) = y(t(1)) = y_0$ THEN	
$y(i+1) = y(i) + \frac{h}{2}\left[f(t(i), y(i)) + f(t(i+1), y(i) + hf(t(i), y(i)))\right]$	$ytemp = y(i) + hf(t(i), y(i))$ AND $y(i+1) = y(i) + \frac{h}{2}\left[f(t(i), y(i)) + f(t(i+1), ytemp)\right]$
for $i = 1, 2, \ldots, N$	

The Fourth-Order Runge-Kutta Method (RK4)

$$
\begin{aligned}
k_1 &= hf(t_n, y_n) \\
k_2 &= hf(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\
k_3 &= hf(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2) \\
k_4 &= hf(t_n + h, y_n + k_3) \\
\boldsymbol{y_{n+1}} &= \boldsymbol{y_n + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4}
\end{aligned}
$$

$\boxed{\textbf{EXAMPLE 10}}$: Solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1, 2]$, $y(1) = 12$ using $N = 5$ subintervals (so $h = 0.2$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

EXAMPLE 10: Solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 5$ subintervals (so $h = 0.2$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

i	TIME	Y_i (EULER)	$y(t_i)$ (EXACT)	ERROR
0	1.000000	12.000000	12.000000	0.0000000000
1	1.200000	10.800000	10.954451	0.1544511501
2	1.400000	9.900000	10.141851	0.2418510567
3	1.600000	9.192857	9.486833	0.2939758376
4	1.800000	8.618304	8.944272	0.3259683386
5	2.000000	8.139509	8.485281	0.3457724457

i	TIME	Y_i (HEUN)	$y(t_i)$ (EXACT)	ERROR
0	1.000000	12.000000	12.000000	0.0000000000
1	1.200000	10.950000	10.954451	0.0044511501
2	1.400000	10.135268	10.141851	0.0065831996
3	1.600000	9.479190	9.486833	0.0076427302
4	1.800000	8.936112	8.944272	0.0081602678
5	2.000000	8.476895	8.485281	0.0083865803

i	TIME	Y_i (RK4)	$y(t_i)$ (EXACT)	ERROR
0	1.000000	12.000000	12.000000	0.0000000000
1	1.200000	10.954442	10.954451	0.0000090013
2	1.400000	10.141839	10.141851	0.0000119113
3	1.600000	9.486820	9.486833	0.0000127666
4	1.800000	8.944259	8.944272	0.0000128513
5	2.000000	8.485269	8.485281	0.0000126333

EXAMPLE 10: Solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1, 2]$, $y(1) = 12$ using $N = 5$ subintervals (so $h = 0.2$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

i	TIME	Y_i (EULER)	$y(t_i)$ (EXACT)	ERROR
0	1.000000	12.000000	12.000000	0.0000000000
1	1.200000	10.800000	10.954451	0.1544511501
2	1.400000	9.900000	10.141851	0.2418510567
3	1.600000	9.192857	9.486833	0.2939758376
4	1.800000	8.618304	8.944272	0.3259683386
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5	2.000000	8.476895	8.485281	0.0083865803

i	TIME	Y_i (RK4)	$y(t_i)$ (EXACT)	ERROR
0	1.000000	12.000000	12.000000	0.0000000000
1	1.200000	10.954442	10.954451	0.0000090013
2	1.400000	10.141839	10.141851	0.0000119113
3	1.600000	9.486820	9.486833	0.0000127666
4	1.800000	8.944259	8.944272	0.0000128513
5	2.000000	8.485269	8.485281	0.0000126333

Compare the errors at the final time with the errors in the following slides where the same problem is done with $N = 10 \Rightarrow h = 0.1$.

Reminder: solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

Reminder: solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

i	**TIME**	Y_i **(EULER)**	$y(t_i)$ **(EXACT)**	**ERROR**
0	1.000000	12.000000	12.000000	0.0000000000
1	1.100000	11.400000	11.441551	0.0415510709
2	1.200000	10.881818	10.954451	0.0726329683
3	1.300000	10.428409	10.524696	0.0962871408
4	1.400000	10.027316	10.141851	0.1145346232
5	1.500000	9.669198	9.797959	0.1287609816
6	1.600000	9.346891	9.486833	0.1399415906
7	1.700000	9.054801	9.203580	0.1487788322
8	1.800000	8.788483	8.944272	0.1557885535
9	1.900000	8.544359	8.705715	0.1613561825
10	2.000000	8.319507	8.485281	0.1657741033

Reminder: solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

Reminder: solving $\frac{dy}{dt} = -\frac{1}{2t}y$, $t \in [1, 2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

i	**TIME**	Y_i **(HEUN)**	$y(t_i)$ **(EXACT)**	**ERROR**
0	1.000000	12.000000	12.000000	0.0000000000
1	1.100000	11.440909	11.441551	0.0006419800
2	1.200000	10.953370	10.954451	0.0010807989
3	1.300000	10.523310	10.524696	0.0013860946
4	1.400000	10.140250	10.141851	0.0016009727
5	1.500000	9.796206	9.797959	0.0017530864
6	1.600000	9.484972	9.486833	0.0018607202
7	1.700000	9.201644	9.203580	0.0019362809
8	1.800000	8.942284	8.944272	0.0019883768
9	1.900000	8.703692	8.705715	0.0020230975
10	2.000000	8.483237	8.485281	0.0020448245

Reminder: solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

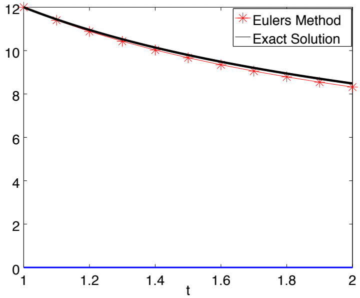
Reminder: solving $\frac{dy}{dt} = -\frac{1}{2t}y$, $t \in [1, 2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

i	**TIME**	Y_i **(RK4)**	$y(t_i)$ **(EXACT)**	**ERROR**
0	1.000000	12.000000	12.000000	0.0000000000
1	1.100000	11.441551	11.441551	0.0000003598
2	1.200000	10.954451	10.954451	0.0000005624
3	1.300000	10.524696	10.524696	0.0000006781
4	1.400000	10.141850	10.141851	0.0000007436
5	1.500000	9.797958	9.797959	0.0000007792
6	1.600000	9.486832	9.486833	0.0000007967
7	1.700000	9.203579	9.203580	0.0000008028
8	1.800000	8.944271	8.944272	0.0000008018
9	1.900000	8.705714	8.705715	0.0000007963
10	2.000000	8.485281	8.485281	0.0000007881

Reminder: solving $\frac{dy}{dt} = -\frac{1}{2t}y$, $t \in [1, 2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

Reminder: solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

- Only an Euler's method plot is shown below since the other methods are sufficiently accurate for their graphs to be indistinguishable from the exact solution graph.



- Note how with Euler's method *(Error $O(h)$)* halving of the step size causes the error in the final step to reduce by about one half, with Heun's method *(Error $O(h^2)$)* the reduction in error is by about one quarter, and with the 4th order Runge-Kutta method *(Error $O(h^4)$)* the reduction in error is by about one sixteenth.

  Students familiar with big O notation should know why.

HOMEWORK:

- Review the material in this lecture and do Tutorial 2. Come to class with questions about any aspect of this review which you find particularly difficult.
- See Tutorial 1 for a quick introduction to Matlab. Since it will be used as a tool in this course, it is useful to have some familiarity with it.
- Have good calculus and linear algebra books nearby.

APPENDIX - Vocabulary/Notation

$\hookrightarrow$ Becoming familiar with the relevant vocabulary and notation is crucial to developing proficiency in mathematics and its applications. Test yourself to see if you have a good understanding of the following, all mentioned/defined earlier in this lecture (and in no particular order):

- Initial value problem.
- Ordinary differential equation.
- Equilibrium solution.
- Separable ODE.
- Stable, unstable, semistable equilibrium solution.
- Partial differential equation.
- First order ODE.
- Direction field.
- Critical point.
- Autonomous first order ODE.
- Separation of variables.
- Linear first order ODE.
- Equilibrium point.
- Nonlinear ODE.