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- I want you to acquire a good understanding of the three approaches but also don't focus exclusively on them; learn to regard them as tools to help you with exploring solutions to the systems of ODEs which arise in your models.

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- For specific models, with initial conditions, often the numerical approach will be invaluable.

Advice On Navigating This Massive (almost 150 pages) Lecture

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- ↪ Otherwise, the main things to try to understand are: how to solve constant coefficient homogeneous systems of linear ODEs, how to convert higher order ODEs to systems of first order ODEs, how to solve IVPs numerically using something like MATLAB's `ode45()`, and how to classify steady states of linear and nonlinear systems of ODEs *(including some intuitive idea of what those steady states look like geometrically and how to interpret them in a real-world application)*.

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$\boxed{\textbf{End of Section}}$

Analytical Solutions to Systems of First Order ODEs

- Much of what we will do next will be similar to what one does to solve linear, constant coefficient ODEs: - for example, the linear second order constant coefficient ODE

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = Q(t)$$

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- We will also be using many of the ideas from the *Supplementary Lecture on Eigenvalues/Eigenvectors.*

Definitions and Conventions

DEFINITIONS: A general system of n first order linear differential equations is one which can be written in the form

$$\begin{aligned} x_1'(t) &= p_{11}(t)x_1(t) + p_{12}(t)x_2(t) + \ldots + p_{1n}(t)x_n(t) + g_1(t) \\ x_2'(t) &= p_{21}(t)x_1(t) + p_{22}(t)x_2(t) + \ldots + p_{2n}(t)x_n(t) + g_2(t) \\ \vdots &= \vdots \\ x_n'(t) &= p_{n1}(t)x_1(t) + p_{n2}(t)x_2(t) + \ldots + p_{nn}(t)x_n(t) + g_n(t) \end{aligned}$$

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or, in matrix form, $\vec{\boldsymbol{x}}\,'(\boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{t})\vec{\boldsymbol{x}}(\boldsymbol{t}) + \vec{\boldsymbol{g}}(\boldsymbol{t})$, where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$,

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So the linear system in EXAMPLE 1 above is *NOT* homogeneous, but

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$$\begin{aligned} 3\frac{dx}{dt} + 3\frac{dy}{dt} - 2x &= e^t \\ \frac{dx}{dt} + 2\frac{dy}{dt} - \ y &= 1 \end{aligned}$$

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$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} -3 & 0 & 17 \\ -4 & 1 & 2 \\ 2 & 2 & -25 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

(as does the "simplified" form of the system in EXAMPLE 2, even though it is not homogeneous).

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We then automatically have that

$$y_1' \quad = \quad y_2$$

Converting n^{th} order Linear ODEs to Linear Systems of First Order ODEs

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- We can now solve for y_1, y_2, y_3, and y_4 and use Equations (1) to convert back to a solution in terms of $u(t)$.

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$\hookrightarrow$ And the matrix form of this system is $\vec{y}\,'(t) = P(t)\vec{y}$, or equivalently

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- ASIDE - if that last problem had initial conditions, *e.g.*, $w(t_0) = a$, $w'(t_0) = b$, and $w''(t_0) = c$, they would become $\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Some General Theory of Linear Systems of First Order ODEs

$$\boxed{\dot{\vec{x}}(t) = P(t)\vec{x}(t) + \vec{g}(t)}$$

$\hookrightarrow$ As with n^{th} order linear single ODEs, we can find a *general solution* to the **inhomogeneous** system of n first order linear ODEs $\dot{\vec{x}}(t) = P(t)\vec{x}(t) + \vec{g}(t)$ by

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1. finding a *general solution to the* ***homogeneous*** *system* $\dot{\vec{x}}(t) = P(t)\vec{x}(t)$,
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- Partial Proof: I'll leave you to show that if $\vec{x_c}(t)$ is a solution to $\dot{\vec{x}}(t) = P(t)\vec{x}(t)$ and $\vec{x_p}(t)$ is a solution to $\dot{\vec{x}}(t) = P(t)\vec{x}(t) + \vec{g}(t)$, then $\vec{x_c}(t) + \vec{x_p}(t)$ is a solution to $\dot{\vec{x}}(t) = P(t)\vec{x}(t) + \vec{g}(t)$.

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2. then finding *ANY particular solution to the full system* $\dot{\vec{x}}(t) = P(t)\vec{x}(t) + \vec{g}(t)$ (using a systems version of the Method of Undetermined Coefficients or other techniques to be discussed later), and
3. adding the two solutions from steps (1) and (2).

- Partial Proof: I'll leave you to show that if $\vec{x_c}(t)$ is a solution to $\dot{\vec{x}}(t) = P(t)\vec{x}(t)$ and $\vec{x_p}(t)$ is a solution to $\dot{\vec{x}}(t) = P(t)\vec{x}(t) + \vec{g}(t)$, then $\vec{x_c}(t) + \vec{x_p}(t)$ is a solution to $\dot{\vec{x}}(t) = P(t)\vec{x}(t) + \vec{g}(t)$.

Therefore, as with n^{th} order single equations, we will concentrate first on finding *general solutions* to the **homogeneous system**, $\dot{\vec{x}}(t) = P(t)\vec{x}(t)$, then we will spend (a little) time learning how to find particular solutions to the **non-homogeneous system**, $\dot{\vec{x}}(t) = P(t)\vec{x}(t) + \vec{g}(t)$ - at least, in the special case of a *constant* matrix $P(t) = A$.

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↦ In other words, if we have n solutions to $\vec{x}\,' = P(t)\vec{x}$ on the interval $\alpha < t < \beta$, we need only evaluate the Wronskian of those n solution vector functions at **ONE** point in the interval $\alpha < t < \beta$ to find out if they are linearly independent or linearly dependent.

- In the approach that we will use for solving the homogeneous system $\vec{x}\,' = A\vec{x}$ (where $A_{n\times n}$ is a constant matrix), we will ensure that the n solutions we obtain are **linearly independent** from the outset - and hence that their linear combination forms a *general solution* to the system $\vec{x}\,' = A\vec{x}$.

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- So there will be no real need to check the linear independence of our solution vector functions using the Wronskian.

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- $\hookrightarrow$ Recall more generally for ODEs such as $ax'' + bx' + cx = 0$ (and higher order linear constant coefficient homogeneous ODEs), we assumed solutions of the form $x(t) = ce^{rt}$ and used a *method of undetermined coefficients approach* to determine what appropriate values for r were (treating c as an arbitrary constant which could be different for different values of r).

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  - As always, we proceed by differentiating $\vec{x}(t) = \vec{c}e^{rt}$ and substituting into the ODE system $\vec{x}\,' = A\vec{x}$.

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*NOTE it does not matter if there are repeated eigenvalues, i.e. $r_i = r_j$ for some $1 \le i < j \le n$, PROVIDED there are n **linearly independent** eigenvectors (see also EXAMPLES 4, 5, 6 and 10 of the Supplementary Lecture on Eigenvalues/Eigenvectors).*

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- So, for example, if $A_{n\times n}$ has n different eigenvalues, then we easily get a general solution of the form of Equation (2).

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Finding Eigenvalues and Eigenvectors Using Matlab

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$>>$ [P, D] = eig(A) returns a diagonal matrix D *(or whatever else you want to call it)* with the eigenvalues of A on its main diagonal, and an invertible matrix P *(or whatever else you want to call it)* whose columns are normalised (scaled so that have length 1) eigenvectors of A in the same order as the corresponding eigenvalues in D. Thus $P^{-1}AP = D$.

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- Type `help eig` in the Matlab command window for more information on eig().

CASE 1: Solving $\vec{x}\,' = A\vec{x}$, $A_{n\times n}$ with n different eigenvalues

$\hookrightarrow$ EXAMPLE 5 Consider the (modified form of the) homogeneous system from EXAMPLE 2, $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4/3 & -1 \\ -2/3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Find a general solution.

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So $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = 2$ are the two (different) eigenvalues of $A = \begin{bmatrix} 4/3 & -1 \\ -2/3 & 1 \end{bmatrix}$.

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$\hookrightarrow$ $\boxed{\text{EXAMPLE 6}}$ Recall from EXAMPLE 2 of the *Supplementary Lecture on Eigenvalues/Eigenvectors* that $A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$, has eigenvalue $\lambda_1 = 7$ with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and eigenvalue $\lambda_2 = -4$ with corresponding eigenvector $\begin{bmatrix} -5 \\ 6 \end{bmatrix}$.
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- $\boxed{\text{ANSWER:}}$ $\vec{x}(t) = B_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + B_2 \begin{bmatrix} -5 \\ 6 \end{bmatrix} e^{-4t}$, where B_1 and B_2 are arbitrary constants.

$\hookrightarrow$ $\boxed{\text{EXAMPLE 7}}$ Similarly, from EXAMPLE 3 of the *Supplementary Lecture on Eigenvalues/Eigenvectors*, since for $A = \begin{bmatrix} 2 & 0 & 0 \\ -4 & -5 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ we have eigenvalues $2, -5$, and 4 with corresponding eigenvectors $\begin{bmatrix} -14 \\ 8 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ respectively, then a general solution to $\dot{\vec{x}} = A\vec{x}$ is

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- NOTE since the coefficient matrix A in $\dot{\vec{x}} = A\vec{x}$ is (lower) triangular, we can also solve this system by solving for $x_1(t)$, then substituting that into the second equation and solving for $x_2(t)$, then substituting those two solutions into the third equation and solving for $x_3(t)$.

$\hookrightarrow$ $\boxed{\text{EXAMPLE 8}}$ Solve the system $\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

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Although this approach is equivalent to the one we have been using so far, it has the advantage of being very useful when solving NON-HOMOGENEOUS problems (so we don't have to use the Method of Undetermined Coefficients or other approaches).

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↪ Having solved for $P^{-1}\vec{x}$ and *written the solution in vector form*, then to find the solution, $\vec{x}$, to the original problem we simply multiply $P^{-1}\vec{x}$ on the left by P.

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CASE 2: Solving $\vec{x}\,' = A\vec{x}$, $A_{n\times n}$ with complex eigenvalues

(but still n linearly independent eigenvectors)

$\hookrightarrow$ Approach will be similar to how we extract real valued (linearly independent) solutions to $ax'' + bx' + cx = 0$ when the characteristic polynomial $ar^2 + br + c$ had (non-real) complex roots *(ignore this comment if you have not solved second order constant coefficient linear differential equations before)*.

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- First observe that if A is a matrix with real entries, then when finding eigenvalues the characteristic equation $\det(A - \lambda \mathrm{I}) = 0$ has only real coefficients so all (non-real) complex eigenvalues occur in *conjugate pairs*: $\lambda = a \pm bi$.

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It is not difficult to show that the corresponding eigenvectors also occur in conjugate pairs.

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$\boxed{\text{PROOF:}}$ $A\vec{x} = (a + ib)\vec{x}$ so taking the complex conjugate of both sides of this equation (recalling that A has only real entries so $\overline{A} = A$ and that $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$), then $A\overline{\vec{x}} = (a - ib)\overline{\vec{x}}$.

$\hookrightarrow$ So let's see how to extract real solutions from solutions which involve eigenvalue $\lambda_1 = a + ib$ with corresponding eigenvector $\vec{u} + i\vec{v}$ *(where $\vec{u}$ and $\vec{v}$ have real entries only)* and the complex conjugate eigenvalue $\lambda_2 = a - ib$ with corresponding eigenvector $\vec{u} - i\vec{v}$.

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- ▶ Recall that in general if r is an eigenvalue of A and $\vec{c}$ is the corresponding eigenvector, then $\vec{c}e^{rt}$ is a solution to $\vec{x}\,' = A\vec{x}$.
  Applying that in this case, we have (complex-valued) solutions $(\vec{u} + i\vec{v})e^{at+ibt}$ and $(\vec{u} - i\vec{v})e^{at-ibt}$.
- ⇝ Clearly, since both complex-valued vector functions above are solutions to $\vec{x}\,' = A\vec{x}$ *then so are their real and imaginary parts*. Without loss of generality,

$$(\vec{u} + i\vec{v})e^{at+ibt} = (\vec{u} + i\vec{v})(e^{at}\cos bt + ie^{at}\sin bt) =$$

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So the two vector functions $e^{at}(\vec{u}\cos bt - \vec{v}\sin bt)$ and $e^{at}(\vec{u}\sin bt + \vec{v}\cos bt)$ are *real-valued* solutions to $\vec{x}\,' = A\vec{x}$. Furthermore, it can be shown that they are linearly independent (so we have **2** linearly independent eigenvectors to go along with the **2** complex-conjugate eigenvalues $a \pm ib$).

$\hookrightarrow$ $\boxed{\text{SUMMARY}}$ Instead of trying to remember the previous result as a formula, just use the following steps to get 2 linearly independent solutions from two complex conjugate eigenvalues $a \pm bi$ to the matrix A when solving $\vec{x}\,' = A\vec{x}$:

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1. Pick *ONE* of the eigenvalues, $\lambda_1 = a + ib$ or $\lambda_2 = a - ib$, and find the corresponding eigenvector $\vec{v}$ by solving $(\boldsymbol{A} - \lambda \mathrm{I})\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$ for $\vec{v}$.

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3. Write the *complex-valued* solution, $\vec{u}$, from Step 2 in terms of its real and imaginary parts: $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u_1}} + i\vec{\boldsymbol{u_2}}$, where both $\vec{u_1}$ and $\vec{u_2}$ are vectors containing only real-valued entries.

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So, for example, any general solution to $\vec{x}\,' = A\vec{x}$ would include the terms $B_1\vec{u_1} + B_2\vec{u_2}$, where B_1 and B_2 are arbitrary constants.

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REMINDER: General solution

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$\hookrightarrow$ $\boxed{\text{EXAMPLE 12}}$ Find the general solution of $\vec{x}\,' = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}.$$

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CASE 3: Solving $\vec{x}\,' = A\vec{x}$, $A_{n\times n}$ with repeated eigenvalues

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$\hookrightarrow$ So what really concerns us is the case where the *multiplicity* of a repeated eigenvalue of A *(= the number of times that the eigenvalue is a root of A's characteristic polynomial)* is *greater than* the number of linearly independent eigenvectors we can find for that eigenvalue - see, for example, EXAMPLE 10 of the *Supplementary Lecture on Eigenvalues/Eigenvectors*.

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- E.g. Suppose A has eigenvalue of multiplicity TWO λ with only 1 linearly independent family of eigenvectors, with $\vec{v}$ being one of them. So $A\vec{v} = \lambda\vec{v}$.

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- E.g. Suppose A has eigenvalue of multiplicity TWO λ with only 1 linearly independent family of eigenvectors, with $\vec{v}$ being one of them. So $A\vec{v} = \lambda\vec{v}$. In the past, we have assumed a solution to $\boldsymbol{\vec{x}\,' = A\vec{x}}$ to be of the form $e^{\lambda t}\vec{v}$.

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  ANSWER: let the solution $\vec{x}(t)$ take the form $\vec{x}(t) = te^{\lambda t}\vec{u}$, where $\vec{u}$ is a constant vector to be determined.

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So again, we must have $\lambda\vec{u_1} - A\vec{u_1} = \vec{0}$ AND $\vec{u_1} + \lambda\vec{u_2} - A\vec{u_2} = \vec{0}$.

REMINDER: $\lambda\vec{u_1} - A\vec{u_1} = \vec{0}$ AND $\vec{u_1} + \lambda\vec{u_2} - A\vec{u_2} = \vec{0}$.

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  - NOTE 1: that $\vec{u_2}$ will typically contain a sum of vectors, one of which will be a multiple of $\vec{u_1}$. We can ignore that multiple of $\vec{u_1}$ since the term $e^{\lambda t}\vec{u_1}$ would appear elsewhere in a general solution to $\boldsymbol{\vec{x}\,' = A\vec{x}}$.

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  - NOTE 2: we have only discussed the case in which λ is an eigenvalue of A of multiplicity TWO. Other cases are "fairly easily" generalisable from this and will be discussed briefly after the next example.

$\hookrightarrow$ EXAMPLE 14 (a) *(Based on EXAMPLE 10 of the Supplementary Lecture on Eigenvalues/ Eigenvectors)*: Find a general solution of $\vec{x}\,' = A\vec{x}$ where $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

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By solving $\det(A - \lambda \mathrm{I}) = (3-\lambda)^2(4-\lambda) = 0$, we get eigenvalues $\boxed{\lambda_1 = 3 \textit{ (multiplicity TWO)}}$ and $\boxed{\lambda_2 = 4}$. We have already seen in the *Supplementary Lecture on Eigenvalues/Eigenvectors* EXAMPLE 10 that there is only one family of linearly independent eigenvectors corresponding to $\lambda_1 = 3$, of which $\vec{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a representative. So we seek a *generalised eigenvector* by assuming a solution of the form $\vec{x}(t) = te^{3t}\vec{u_1} + e^{3t}\vec{u_2}$ and, upon substitution into $\boldsymbol{\vec{x}\,' = A\vec{x}}$ solving the resulting new equation

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So $u_{23} = 0$, $u_{22} = 1$, and u_{21} can take on any (non-zero) value, so that a typical generalised eigenvector is of the form $u_{21} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

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REMINDER: Generalised eigenvector : $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\hookrightarrow$ Because the first vector in this sum is simply the eigenvector $\vec{u_1}$, we ignore it and take the *generalised eigenvector* to be simply $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

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$$\vec{x}(t) = B_1 e^{4t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + B_2 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + B_3\left(te^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right).$$

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We first find the eigenvalues of A by solving $\det(A - \lambda I) = 0$, so $(3-\lambda)(1-\lambda)+1 = 0 \Rightarrow \lambda^2 - 2\lambda + 4 = 0 \Rightarrow (\lambda-2)^2 = 0$, so that $\boxed{\lambda = 2}$ is the (repeated) eigenvalue.

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To find one or more corresponding linearly independent eigenvectors, solve $(A - 2I)\vec{x} = \vec{0}$: $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2 \text{ so } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or any (non-zero) scalar multiple thereof is the only (family of) linearly independent eigenvector(s) associated directly with the eigenvalue $\lambda = 2$.

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We therefore need to find a *generalised eigenvector* by solving

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow u_1 - u_2 = 1 \text{ or } u_1 = 1 + u_2.$$

So any vector of the form $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 + u_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a generalised eigenvector of A. Specifically, since the second vector already appears in the linear span of the first eigenvector, we take just $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as the *generalised eigenvector* (i.e, we set $u_2 = 0$).

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$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 t e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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$$(A-\lambda I)\vec{w}_1 = 0, \quad (A-\lambda I)\vec{w}_2 = \vec{w}_1, \quad \text{and} \quad \boxed{\boldsymbol{(A-\lambda I)\vec{w}_3 = \vec{w}_2}}.$$

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- SEE a standard introductory ODE book, such as the one by *Boyce and DiPrima*, for more on this topic.

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- Note examples are numbered in Appendix B and the rest of this document as if Appendix B were inserted here.

End of Section

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  - A function whose output is a vector will follow the above convention of having its name in boldface or with a vector symbol: eg $\vec{f}(t,y) = \begin{bmatrix} t^2 y \\ sin(t+2y) \end{bmatrix}$ or $\boldsymbol{g}(t) = \begin{bmatrix} t^2+2t-1 \\ sin(t)e^{-t} \end{bmatrix}$.

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This convention will be useful to adopt when we write general programs (Euler's, Heun's, RK(4), etc.) to solve systems of n first order ODEs.

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$$\begin{aligned}\frac{dy_1}{dt} &= f_1(t, y_1, y_2, y_3)\\ \frac{dy_2}{dt} &= f_2(t, y_1, y_2, y_3)\\ \frac{dy_3}{dt} &= f_3(t, y_1, y_2, y_3)\end{aligned}$$

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- NOTE the similarity to the single first order IVP.
- *Often, $\vec{t}_0$ will be $\vec{0}$, and we will focus on the $n = 2$ and $n = 3$ cases.*

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$\hookrightarrow$ Since many of the systems we look at will also be **autonomous**, here is the **general first order system of n IVPs** for that special case:

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- NOTE the similarity to the single first order **autonomous** IVP *BUT* also NOTE that as for single ODEs *we will solve these systems using programs written for the general case on the preceding slide.*

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I will show next why this is true for Euler's method by deriving the method for the special case of 2 ODEs, using Taylor series of the two solution functions, similar to what we did in **Lecture 2** when deriving Euler's method for single ODEs.

- Suppose we want to use Euler's method to approximate the solutions to

$$\begin{aligned} \frac{dy_1}{dt} &= f_1(t_1, y_1, y_2) \\ \frac{dy_2}{dt} &= f_2(t_1, y_1, y_2), \quad t \in [t_0, T], \quad y_1(t_0) = y_{1,0}, \quad y_2(t_0) = y_{2,0}. \end{aligned}$$

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- Then, using Taylor series,

$$\begin{aligned}y_1(t_{i+1}) = y_1(t_i + h) &= y_1(t_i) + hy_1'(t_i) + O(h^2) \approx y_1(t_i) + hf_1(t_i, y_1(t_i), y_2(t_i))\\ y_2(t_{i+1}) = y_2(t_i + h) &= y_2(t_i) + hy_2'(t_i) + O(h^2) \approx y_2(t_i) + hf_2(t_i, y_1(t_i), y_2(t_i))\end{aligned}$$

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$$\boxed{\begin{gathered} \text{or } \vec{Y}^{(i+1)} = \vec{Y}^{(i)} + h\vec{f}(t_i, Y_1^{(i)}, Y_2^{(i)}) \\ \text{where } \vec{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \text{ and } \vec{f}(t, Y_1, Y_2) = \begin{bmatrix} f_1(t, Y_1, Y_2) \\ f_2(t, Y_1, Y_2) \end{bmatrix} \end{gathered}}$$

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Euler's method for approximating the solution to the *general first order system of n IVPs*, $\dfrac{d\vec{y}}{dt} = \vec{f}(t, \vec{y}), \quad \forall\ t \in [t_0, T], \quad \vec{y}(t_0) = \vec{y}_0$:

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$$\boldsymbol{\vec{Y}_0 = \vec{y}(t_0)} \text{ THEN}$$

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and $Y_i^{(j)}$ is the Euler approximation to $y_i(t_j)$ (for $i = 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, N$).

Reminder: $\vec{Y}^{(i+1)} = \vec{Y}^{(i)} + h\vec{f}(t_i, \vec{Y}^{(i)})$ for $i = 0, 1, 2 \ldots, N-1$

In summary, Euler's method for a first order system of ODEs simply consists of applying the *scalar* Euler's method to a vector of differential equations one component at a time.

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In summary, Euler's method for a first order system of ODEs simply consists of applying the *scalar* Euler's method to a vector of differential equations one component at a time.

- I include in the following pages a sample Euler's method program for a system of two differential equations. Modifying it for a system of 3 or more equations and for Heun's method and the Runge-Kutta (fourth order) methods is relatively straightforward. NOTE a somewhat more sophisticated version will be also provided on the course Moodle page.

```
clear
clf

f = @(t,y) [-4*y(1)-2*y(2) + cos(t) + 4*sin(t);
            3*y(1)+y(2)-3*sin(t)];

% Here we give the exact solution if known. If not known, set to the
% this to return an appropriately-sized vector of zeros and ignore all
% subsequent references to the exact solution in this program
exact = @(t) [2*exp(-t) - 2*exp(-2*t) + sin(t);
                -3*exp(-t) + 2*exp(-2*t)];

n = input('Enter the number of equations in your system of ODEs ');

h=0.1;
t0 = 0; tN = 2;
y0 = [0; -1];

if length(y0) ~= n
    disp('Error, you entered an incorrect number of equations. Try again ')
    return;
end
```

```
t = [t0:h:tN];
sizet = length(t);


% y(i,j) is the approximation to y_i(t_j)
y = zeros(n,sizet);
yexact = zeros(n, sizet);
for (k = 1:n)
     y(k,1) = y0(k);
end

% Main Euler's method loop
for k = 2:sizet
  y(:,k) = y(:,k-1) + h*f(t(k-1), y(:,k-1));
end

for(k = 1:sizet)
  yexact(:,k) = exact(t(k));
end


for (mm = 1:n)
```

```
        fprintf('\nPRINTING INFORMATION FOR FUNCTION %d\n\n',mm);
        fprintf('  i   TIME        Yi (APPROX)     y(ti) (EXACT)      ABS. ERROR\n')
        for k = 1:sizet
fprintf('%3d   %8f   %10f      %10f        %10f\n',k-1,t(k),y(mm,k),yexact(mm,k), abs(y(mm,k)-yexact(mm,k)))
        end
    end

    disp(''); % blank line

    plotsoln=input('Hit return for graphs of solutions versus time ')
    if isempty(plotsoln)
        set(gca,'fontsize',14)
        for k = 1:n
            plot(t,y(k,:),'linewidth',2)
            xlabel('t')
            fprintf('\nPLOTTING INFORMATION FOR FUNCTION %d\n\n',k);
            if k < n
                disp('Hit any key to see the next graph ');
                pause
            end
        end
    end
```

```
disp(' ') % blank line

if n == 2 % phase plane plot
    plotsoln=input('Hit return for phase plane plot ')
    if isempty(plotsoln)
        plot(y(1,:), y(2,:),'-r');
        xlabel('y1')
        ylabel('y2')
    end
end
```

$\hookrightarrow$ EXAMPLE 17 Use the systems Euler's method with $\boxed{h = 0.1}$ to solve

$$\frac{dy_1}{dt} = -4y_1 - 2y_2 + \cos(t) + 4\sin(t)$$

$$\frac{dy_2}{dt} = 3y_1 + y_2 - 3\sin(t)$$

$$t \in [0, 2], \quad y_1(0) = 0,\ y_2(0) = -1$$

(EXACT SOLUTION)

$$y_1(t) = 2e^{-t} - 2e^{-2t} + \sin(t)$$

$$y_2(t) = -3e^{-t} + 2e^{-2t}$$

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$$\begin{aligned}\frac{dy_1}{dt} &= -4y_1 - 2y_2 + \cos(t) + 4\sin(t)\\ \frac{dy_2}{dt} &= 3y_1 + y_2 - 3\sin(t)\\ t \in [0,2], &\quad y_1(0) = 0,\ y_2(0) = -1\end{aligned}$$

(EXACT SOLUTION)

$$y_1(t) = 2e^{-t} - 2e^{-2t} + \sin(t)$$

$$y_2(t) = -3e^{-t} + 2e^{-2t}$$

↪ In vector form, this is $\dfrac{d\vec{y}}{dt} = \vec{f}(t, y_1, y_2), \quad t \in [0,2], \quad \vec{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, where

$\vec{y} =$

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$$y_1(t) = 2e^{-t} - 2e^{-2t} + \sin(t)$$

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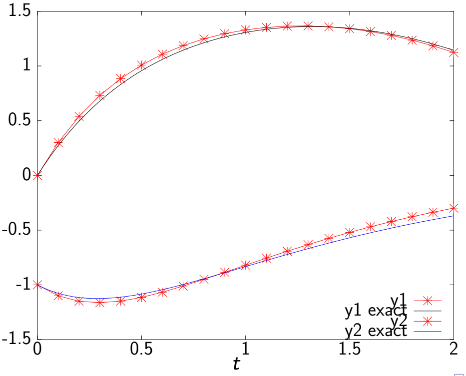
$$\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ and } \vec{f}(t, y_1, y_2) = \begin{bmatrix} -4y_1 - 2y_2 + \cos(t) + 4\sin(t) \\ 3y_1 + y_2 - 3\sin(t) \end{bmatrix}$$

- You will be expected to know how to change easily between the vector and non-vector form of such systems of ODEs.

Reminder: solving $\frac{d\vec{y}}{dt} = \vec{f}(t, y_1, y_2), \quad t \in [0, 2], \quad \vec{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ with

$$\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ and } \vec{f}(t, y_1, y_2) = \begin{bmatrix} -4y_1 - 2y_2 + \cos(t) + 4\sin(t) \\ 3y_1 + y_2 - 3\sin(t) \end{bmatrix}$$

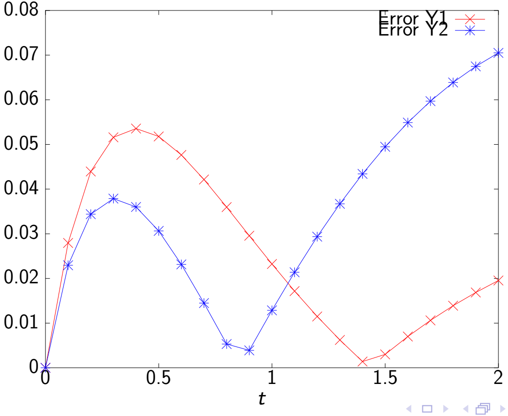
$y_1(t)$ and $y_2(t)$- Exact Solutions and Euler's Method Approximations



Reminder: solving $\frac{d\vec{y}}{dt} = \vec{f}(t, y_1, y_2), \quad t \in [0, 2], \quad \vec{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ with

$$\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ and } \vec{f}(t, y_1, y_2) = \begin{bmatrix} -4y_1 - 2y_2 + \cos(t) + 4\sin(t) \\ 3y_1 + y_2 - 3\sin(t) \end{bmatrix}$$

$y_1(t)$ and $y_2(t)$- Error in Euler's Method Approximations



Y_1

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR
0	0.000000	0.000000	0.000000	0.000000
1	0.100000	0.300000	0.272047	0.027953
2	0.200000	0.539434	0.495491	0.043943
3	0.300000	0.731125	0.679533	0.051591
4	0.400000	0.884960	0.831401	0.053559
5	0.500000	1.008510	0.956728	0.051782
		⋮		
16	1.600000	1.314846	1.321842	0.006996
17	1.700000	1.279670	1.290285	0.010615
18	1.800000	1.235906	1.249798	0.013892
19	1.900000	1.183836	1.200696	0.016860
20	2.000000	1.123791	1.143337	0.019546

Y_2

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR
0	0.000000	−1.000000	−1.000000	0.000000
1	0.100000	−1.100000	−1.077051	0.022949
2	0.200000	−1.149950	−1.115552	0.034398
3	0.300000	−1.162716	−1.124831	0.037884
4	0.400000	−1.148306	−1.112302	0.036004
5	0.500000	−1.114474	−1.083833	0.030641
		⋮		
16	1.600000	−0.469265	−0.524165	0.054900
17	1.700000	−0.421610	−0.481304	0.059694
18	1.800000	−0.377370	−0.441249	0.063880
19	1.900000	−0.336489	−0.403964	0.067475
20	2.000000	−0.298877	−0.369375	0.070497

$\hookrightarrow$ $\boxed{\text{EXAMPLE 18}}$ Use the systems Euler's method to solve $x''' - x' = t$, $t \in [0, 4]$, $x(0) = 6$, $x'(0) = -5$, $x''(0) = 0$. (EXACT SOLUTION, $x(t) = 5 - 2e^t + 3e^{-t} - \frac{1}{2}t^2$).

$\hookrightarrow$ $\boxed{\text{EXAMPLE 18}}$ Use the systems Euler's method to solve $x''' - x' = t$, $t \in [0, 4]$, $x(0) = 6$, $x'(0) = -5$, $x''(0) = 0$. (EXACT SOLUTION, $x(t) = 5 - 2e^t + 3e^{-t} - \frac{1}{2}t^2$).

$\succ$ $\boxed{\text{ANSWER}}$ First we convert it to a system:
Let $y_1 = x, \;\; y_2 = x', \;\;\; y_3 = x'' \;\; \Rightarrow$

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Let $y_1 = x$, $y_2 = x'$, $y_3 = x''$ $\Rightarrow$

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Meanwhile, the initial conditions become

$$\vec{y}(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix}.$$

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(EXACT SOLUTION

$y_1(t) =$

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(EXACT SOLUTION
$\boldsymbol{y_1(t) = 5 - 2e^t + 3e^{-t} - \frac{1}{2}t^2, \quad y_2(t) = -2e^t - 3e^{-t} - t, \quad y_3(t) =}$

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(EXACT SOLUTION
$y_1(t) = 5 - 2e^t + 3e^{-t} - \frac{1}{2}t^2$, $y_2(t) = -2e^t - 3e^{-t} - t$, $y_3(t) = -2e^t + 3e^{-t} - 1$).

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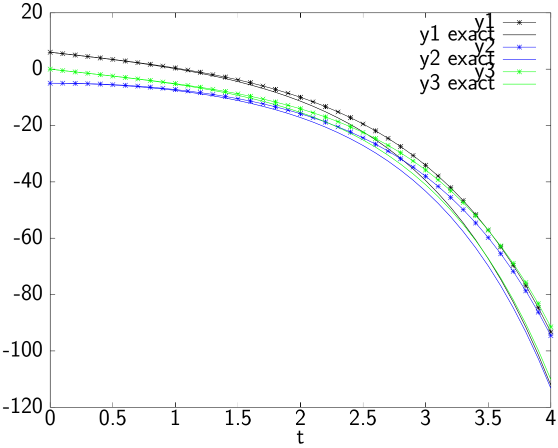
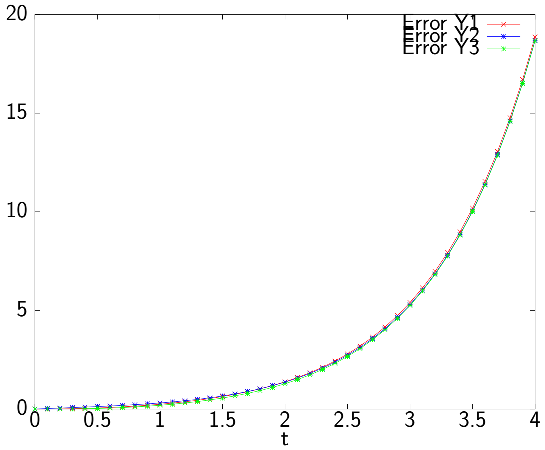
- We now need only make minor modifications to the earlier Euler's method program for a system of 2 equations to get it to work for 3 equations. The results are summarised on the following pages, first for $h = 0.1$ then for $h = 0.01$.

$y_1(t)$, $y_2(t)$, and $y_3(t)$ - Exact Solutions and Euler's Method Approximations with $h = 0.1$



$y_1(t)$, $y_2(t)$, and $y_3(t)$ - Error in Euler's Method Approximations with $h = 0.1$



Y_1

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR
0	0.000000	6.000000	6.000000	0.000000
1	0.100000	5.500000	5.499170	0.000830
2	0.200000	5.000000	4.993387	0.006613
		⋮		
38	3.800000	−76.783943	−91.555257	14.771314
39	3.900000	−84.650286	−101.349172	16.698887
40	4.000000	−93.274168	−112.141353	18.867185

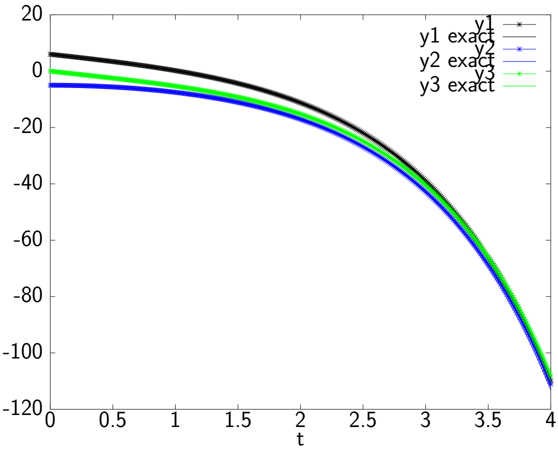
Y_2

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR
0	0.000000	−5.000000	−5.000000	0.000000
1	0.100000	−5.000000	−5.024854	0.024854
2	0.200000	−5.050000	−5.098998	0.048998
		⋮		
38	3.800000	−78.663431	−93.269481	14.606050
39	3.900000	−86.238825	−102.765624	16.526799
40	4.000000	−94.562854	−113.251247	18.688393

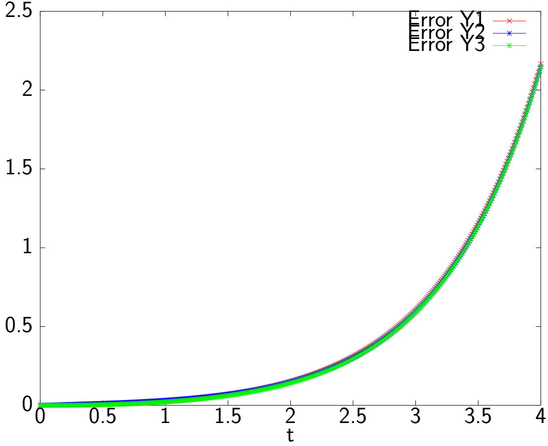
Y_3

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR
0	0.000000	0.000000	0.000000	0.000000
1	0.100000	−0.500000	−0.495830	0.004170
2	0.200000	−0.990000	−0.986613	0.003387
		⋮		
38	3.800000	−75.753943	−90.335257	14.581314
39	3.900000	−83.240286	−99.744172	16.503887
40	4.000000	−91.474168	−110.141353	18.667185

$y_1(t)$, $y_2(t)$, and $y_3(t)$ - Exact Solutions and Euler's Method Approximations with $h = 0.01$



$y_1(t)$, $y_2(t)$, and $y_3(t)$ - Error in Euler's Method Approximations with $h = 0.01$



Y_1

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR
0	0.000000	6.000000	6.000000	0.000000
1	0.010000	5.950000	5.949999	0.000001
2	0.020000	5.900000	5.899993	0.000007
		⋮		
398	3.980000	−107.784316	−109.898212	2.113895
399	3.990000	−108.874055	−111.014330	2.140274
400	4.000000	−109.974383	−112.141353	2.166970

Y_2

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR
0	0.000000	−5.000000	−5.000000	0.000000
1	0.010000	−5.000000	−5.000250	0.000250
2	0.020000	−5.000500	−5.000999	0.000499
		⋮		
398	3.980000	−108.973906	−111.070125	2.096219
399	3.990000	−110.032747	−112.155278	2.122531
400	4.000000	−111.102086	−113.251247	2.149161

Y_3

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR
0	0.000000	0.000000	0.000000	0.000000
1	0.010000	−0.050000	−0.049951	0.000049
2	0.020000	−0.099900	−0.099807	0.000093
		⋮		
398	3.980000	−105.884016	−107.978012	2.093995
399	3.990000	−106.933955	−109.054280	2.120324
400	4.000000	−107.994383	−110.141353	2.146970

Other Numerical Methods for Systems of First Order ODEs

$\hookrightarrow$ The relatively large errors in **EXAMPLE 18** with stepsize $h = 0.1$ are a good reason why we move on now to other *(higher order)* numerical methods for systems of first order IVPs.

Other Numerical Methods for Systems of First Order ODEs

- ↪ The relatively large errors in **EXAMPLE 18** with stepsize $h = 0.1$ are a good reason why we move on now to other *(higher order)* numerical methods for systems of first order IVPs.
- ↪ As mentioned earlier, the equations for the systems version of the different numerical methods remain the same as their scalar counterparts when written in vector notation *(with the appropriate vectorised interpretation of $+, -$ and multiplication by h)*, and applying a numerical IVP method to a system of ODEs simply consists of applying the *scalar* form of that method to a vector of differential equations one component at a time.

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- As mentioned earlier, the equations for the systems version of the different numerical methods remain the same as their scalar counterparts when written in vector notation *(with the appropriate vectorised interpretation of* $+, -$ *and multiplication by h)*, and applying a numerical IVP method to a system of ODEs simply consists of applying the *scalar* form of that method to a vector of differential equations one component at a time.
  - In these notes, we will only look at systems versions of Heun's method and the 4th order Runge-Kutta method.

    (For systems versions of other methods, such as TS(2) and AB(2), you can consult the MATH1106 Lecture Notes [contact me if you do not have access to those notes and wish to see them] or books on numerical solutions to ODEs).

Heun's Method

$\hookrightarrow$ **In summary, Heun's method for a first order system of ODEs simply consists of applying the *scalar* Heun's method to a vector of differential equations one component at a time.**

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$\hookrightarrow$ The *TWO-STEP (see **Lecture 2**)* version of the method is summarised on the following page *for a system of n first order IVPs*.

Heun's Method for Systems

Heun's method for approximating the solution to the *general first order system of n IVPs*, $\dfrac{d\vec{y}}{dt} = \vec{f}(t, \vec{y}), \quad \forall\ t \in [t_0, T], \quad \vec{y}(t_0) = \vec{y}_0$:

Heun's Method for Systems

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Heuns's Method for Systems

$$\overrightarrow{Y}_0 = \vec{y}(t_0) \text{ THEN}$$

$$\overrightarrow{Ytemp}^{(i+1)} = \overrightarrow{Y}^{(i)} + h\vec{f}(t_i, \overrightarrow{Y}^{(i)}) \text{ AND}$$

$$\overrightarrow{Y}^{(i+1)} = \overrightarrow{Y}^{(i)} + \frac{h}{2}\left[\vec{f}(t_i, \overrightarrow{Y}^{(i)}) + f(t_{i+1}, \overrightarrow{Ytemp}^{(i+1)}\right] \text{ for } i = 0, 1, 2 \ldots, N-1.$$

Heun's Method for Systems

Heun's method for approximating the solution to the *general first order system of n IVPs*, $\dfrac{d\vec{y}}{dt} = \vec{f}(t, \vec{y}), \quad \forall\ t \in [t_0, T], \quad \vec{y}(t_0) = \vec{y}_0$:

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$$\vec{Y}^{(i+1)} = \vec{Y}^{(i)} + \frac{h}{2}\left[\vec{f}(t_i, \vec{Y}^{(i)}) + f(t_{i+1}, \overrightarrow{Ytemp}^{(i+1)})\right] \text{ for } i = 0, 1, 2 \ldots, N-1.$$

where

$$\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \text{ and } \vec{f}(t, \vec{Y}) = \vec{f}(t, Y_1, Y_2, \ldots, Y_n) = \begin{bmatrix} f_1(t, Y_1, Y_2, \ldots, Y_n) \\ f_2(t, Y_1, Y_2, \ldots, Y_n) \\ \vdots \\ f_n(t, Y_1, Y_2, \ldots, Y_n) \end{bmatrix}$$

and $Y_i^{(j)}$ is the Heun approximation to $y_i(t_j)$ (for $i = 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, N$).

$\hookrightarrow$ $\boxed{\text{EXAMPLE 19}}$ - Redo **EXAMPLE 17** using Heun's method: *use the systems Heun's method with* $\boxed{h = 0.1}$ *to solve*

$$\frac{dy_1}{dt} = -4y_1 - 2y_2 + \cos(t) + 4\sin(t)$$

$$\frac{dy_2}{dt} = 3y_1 + y_2 - 3\sin(t)$$

$$t \in [0, 2], \quad y_1(0) = 0,\ y_2(0) = -1$$

(EXACT SOLUTION)

$$y_1(t) = 2e^{-t} - 2e^{-2t} + \sin(t)$$

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$$y_1(t) = 2e^{-t} - 2e^{-2t} + \sin(t)$$
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↪ Recall that in vector form, this is

$$\frac{d\vec{y}}{dt} = \vec{f}(t, y_1, y_2), \quad t \in [0,2], \quad \vec{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \text{ where}$$

$$\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ and } \vec{f}(t, y_1, y_2) = \begin{bmatrix} -4y_1 - 2y_2 + \cos(t) + 4\sin(t) \\ 3y_1 + y_2 - 3\sin(t) \end{bmatrix}.$$

Reminder: solving $\frac{d\vec{y}}{dt} = \vec{f}(t, y_1, y_2)$, $\quad t \in [0, 2]$, $\quad \vec{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ with

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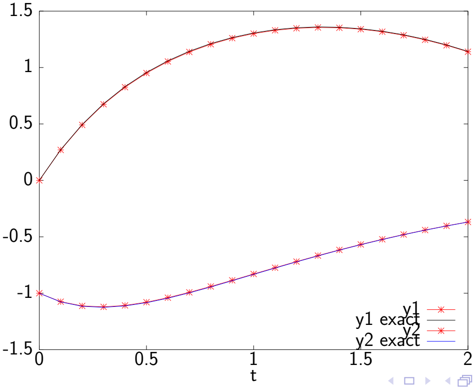
$y_1(t)$ and $y_2(t)$- Exact Solutions and Heun's Method Approximations



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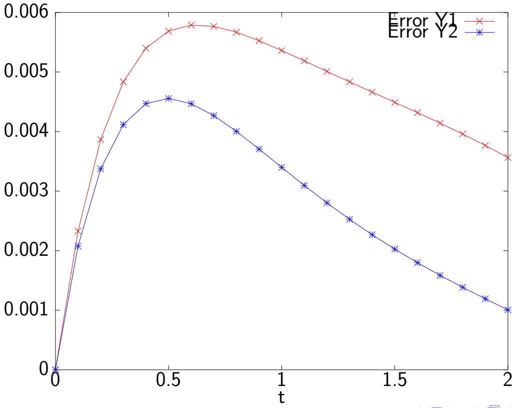

Y_1

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR	Euler's Error
0	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.100000	0.269717	0.272047	0.002330	0.027953
2	0.200000	0.491624	0.495491	0.003867	0.043943
3	0.300000	0.674699	0.679533	0.004834	0.051591
4	0.400000	0.826001	0.831401	0.005400	0.053559
5	0.500000	0.951041	0.956728	0.005687	0.051782
			⋮		
16	1.600000	1.317525	1.321842	0.004318	0.006996
17	1.700000	1.286144	1.290285	0.004141	0.010615
18	1.800000	1.245840	1.249798	0.003958	0.013892
19	1.900000	1.196929	1.200696	0.003766	0.016860
20	2.000000	1.139774	1.143337	0.003563	0.019546

Y_2

i	TIME	Y_i (APPROX)	$y(t_i)$ (EXACT)	ABS. ERROR	Euler's Error
0	0.000000	−1.000000	−1.000000	0.000000	0.000000
1	0.100000	−1.074975	−1.077051	0.002076	0.022949
2	0.200000	−1.112178	−1.115552	0.003374	0.034398
3	0.300000	−1.120715	−1.124831	0.004117	0.037884
4	0.400000	−1.107832	−1.112302	0.004470	0.036004
5	0.500000	−1.079276	−1.083833	0.004557	0.030641
			⋮		
16	1.600000	−0.522366	−0.524165	0.001799	0.054900
17	1.700000	−0.479718	−0.481304	0.001586	0.059694
18	1.800000	−0.439865	−0.441249	0.001385	0.063880
19	1.900000	−0.402772	−0.403964	0.001192	0.067475
20	2.000000	−0.368369	−0.369375	0.001006	0.070497

RK4 Method

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$\hookrightarrow$ You should try to implement this for two equations by modifying the earlier Euler's or Heun's method program; this is easier if you write it out in vector form and then think of how to update the components of those vectors.

RK4 Method for Systems

$$\vec{Y}_0 = \vec{y}(t_0) \text{ THEN}$$

$$\begin{aligned}
\vec{k}_1 &= h\vec{f}(t_i, \vec{Y}^{(i)}) \\
\vec{k}_2 &= hf(t_i + \tfrac{1}{2}h, \vec{Y}^{(i)} + \tfrac{1}{2}\vec{k}_1) \\
\vec{k}_3 &= hf(t_i + \tfrac{1}{2}h, \vec{Y}^{(i)} + \tfrac{1}{2}\vec{k}_2) \\
\vec{k}_4 &= hf(t_i + h, \vec{Y}^{(i)} + \vec{k}_3)
\end{aligned}$$

AND

$$\vec{Y}^{(i+1)} = \vec{Y}^{(i)} + \tfrac{1}{6}\vec{k}_1 + \tfrac{1}{3}\vec{k}_2 + \tfrac{1}{3}\vec{k}_3 + \tfrac{1}{6}\vec{k}_4$$

for $i = 0, 1, 2 \ldots, N-1$

where

$$\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \quad \text{and} \quad \vec{f}(t, \vec{Y}) = \vec{f}(t, Y_1, Y_2, \ldots, Y_n) = \begin{bmatrix} f_1(t, Y_1, Y_2, \ldots, Y_n) \\ f_2(t, Y_1, Y_2, \ldots, Y_n) \\ \vdots \\ f_n(t, Y_1, Y_2, \ldots, Y_n) \end{bmatrix}$$

and $Y_i^{(j)}$ is the RK4 approximation to $y_i(t_j)$ (for $i = 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, N$).

$\hookrightarrow$ EXAMPLE 20 - Redo **EXAMPLES 17 & 19** using the RK4 method: *use the systems RK4 method with* $\boxed{h = 0.1}$ *to solve*

$$\frac{dy_1}{dt} = -4y_1 - 2y_2 + \cos(t) + 4\sin(t)$$

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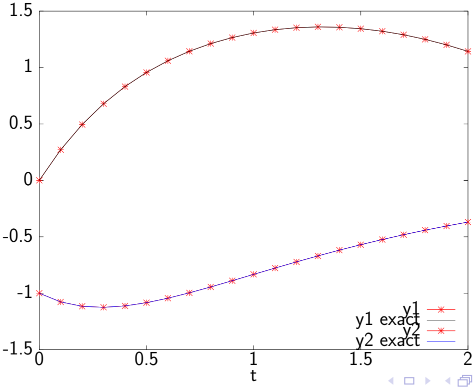
$y_1(t)$ and $y_2(t)$- Exact Solutions and RK4 Method Approximations



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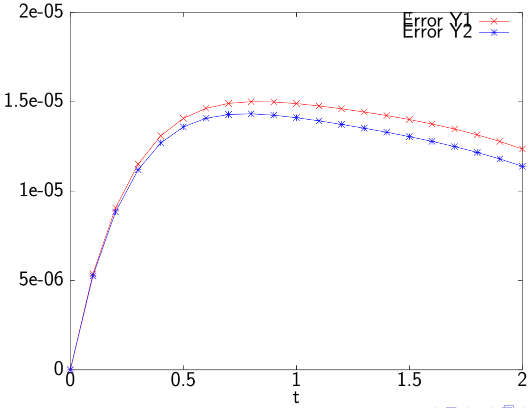

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End of Section

Geometrical Study of Solutions to Systems of First Order ODEs

Introduction

$\hookrightarrow$ More so than for single differential equations, systems of differential equations are hard to solve (in the earlier part of this lecture we only considered a very small subset: linear homogeneous systems with constant coefficient matrices). Furthermore, often we are just interested in *patterns* or *general behaviour* of solutions to the scenario being modelled by a system of ODEs, *and these can be determined by a geometric analysis of the system of ODEs without solving it!*.

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↪ Again, much of the work will be done initially for *linear constant coefficient systems of ODEs*, and then the very straightforward generalisation to nonlinear systems will be covered.

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- Sometimes information from number 3 is used to help inform the creation of phase portraits in number 2 and/or to help choose a suitable domain in which to generate a direction field in number 1.

Phase Space, Phase Portraits, Direction Fields, Steady States - Vocabulary

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↦ All of the subsequent definitions of key terminology such a *nullclines* and *steady states* will be related to/based on the graphs of solutions in *phase space*.

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$$\frac{d\vec{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \quad with \quad \vec{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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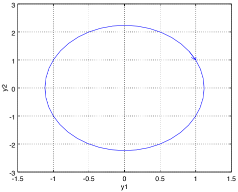
along with plots of y_1 and y_2 versus t and a description of how one could get from the phase plane plot to the solutions versus time plots.

Phase Plane Plot



y_i Versus t Plot

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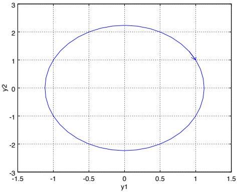

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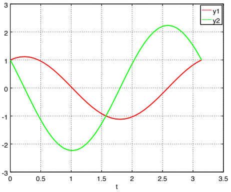

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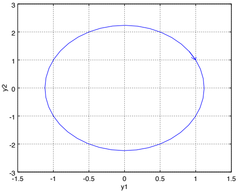

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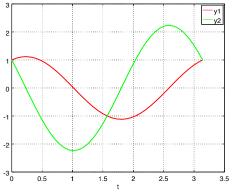

- Starting at $(y_1, y_2) = (1, 1)$ in the phase plane, we see that the value of y_1 increases briefly to its peak slightly above 1 then decreases all the way down to a value slightly below -1 then increases all the way to 1 again.

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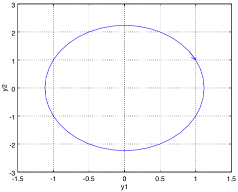

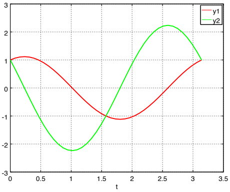

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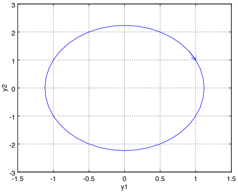
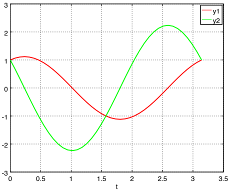


y_i Versus t Plot



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- These two solution behaviours as functions of time are confirmed by the second plot above.

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- Thus the only steady state of (nonsingular) linear systems of ODEs is the origin in phase space.

- $\boxed{\text{EXAMPLE 22}}$: Find the steady state(s) of $\dfrac{d\vec{y}}{dt} = \begin{pmatrix} 6y_1 + 6y_2 + 12 \\ 6y_1 + 6y_2^2 - 24 \end{pmatrix}$.
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- I'd suggest plotting the two graphs together to see approximately where they intersect and then choosing starting values close to each intersection point.

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- In the following diagrams, the origin $\mathbf{(0,0)}$, is the steady state.

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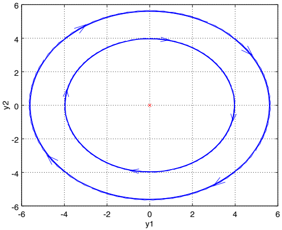
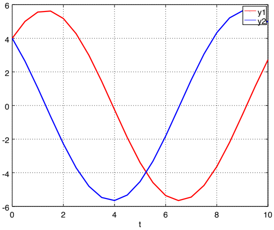


Phase Plane Plot



Sample Solution vs Time Plot

$\hookrightarrow$ The choice of the name *centre* is obvious from the phase plane plot.

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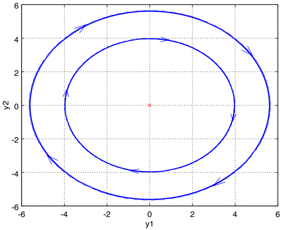

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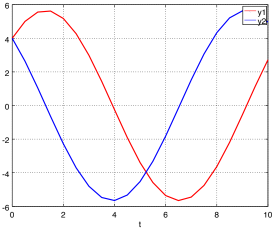

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↪ The solutions are periodic functions which each oscillate around its component of the steady state.

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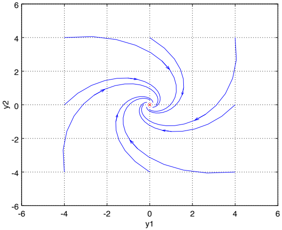

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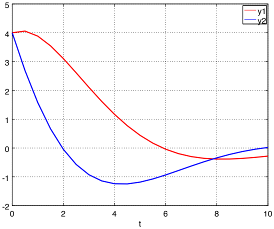

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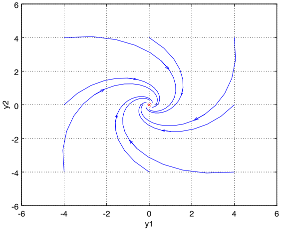

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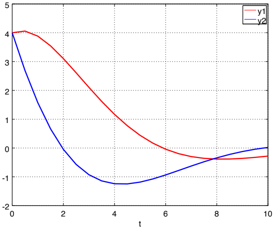

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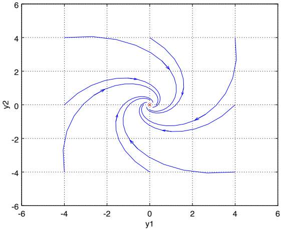
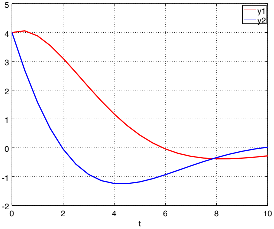
- The choice of the name *spiral point* or *focus* is obvious from the phase plane plot.
- This shows an asymptotically stable spiral point. In an unstable spiral point, the arrows would point in the opposite direction (away from the origin).

Spiral Point/Focus



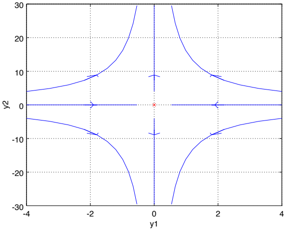
Phase Plane Plot



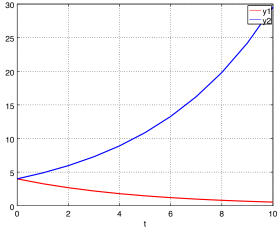
Sample Solution vs Time Plot

- ↪ The choice of the name *spiral point* or *focus* is obvious from the phase plane plot.
- ↪ This shows an asymptotically stable spiral point. In an unstable spiral point, the arrows would point in the opposite direction (away from the origin).
- ↪ The sample solutions for an asymptotically stable spiral point are shown. Each function oscillates around its component of the steady state and the amplitude of those oscillations decrease with increasing time. For an unstable spiral point, the oscillations would increase with increasing time.

Saddle Point



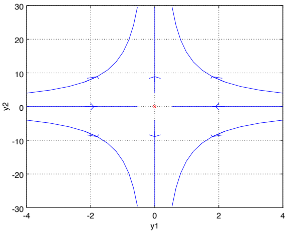
Phase Plane Plot



Sample Solution vs Time Plot

$\hookrightarrow$ The choice of the name *saddle point* is obvious from the phase plane plot if one remembers that term from optimisation of functions of 2 independent variables.

Saddle Point



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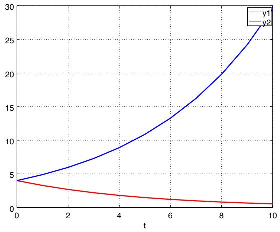

Sample Solution vs Time Plot

- ↪ The choice of the name *saddle point* is obvious from the phase plane plot if one remembers that term from optimisation of functions of 2 independent variables.
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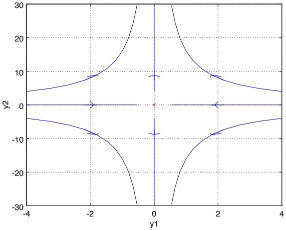

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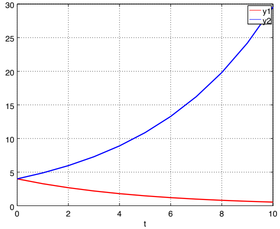

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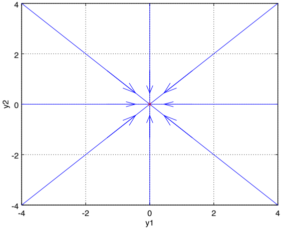
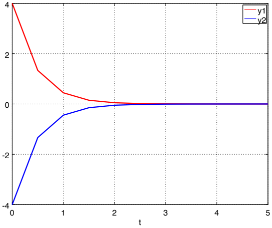
- ↪ So saddle points are always unstable.

Node





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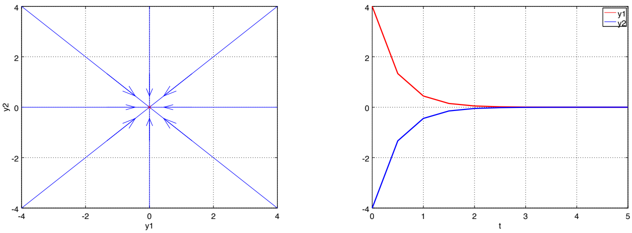
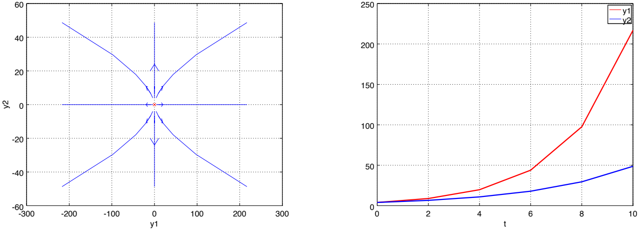
$\boxed{\text{Node}}$

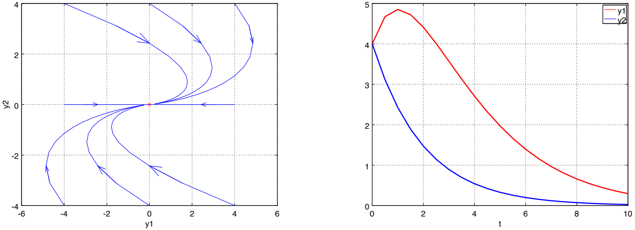


- ↪ This shows a proper node. The plots on the next page show improper nodes.
- ↪ This node is asymptotically stable (a sink). On the following page, one node is unstable and the other is asymptotically stable. Obviously, reversing the arrows on the diagrams changes an asymptotically stable node to unstable and vice versa.

Node - continued



An Unstable *Improper* Node



An Asymptotically Stable *Improper* Node

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↪ In particular, at each point t, the vector $\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}\right)$ is *tangent* to the trajectory traced out by $(y_1(t), y_2(t))$ in the *phase plane*. Ask if you are not sure why. *(Technically, at each point t the position vector $\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}\right)$, with origin at $(0, 0)$, when shifted to the point (y_1, y_2) in the phase plane is tangent to the curve traced out by $(y_1(t), y_2(t))$).*

$\hookrightarrow$ So if on a grid of $(y_1,\ y_2)$ values we plot a little line segment parallel to $(F_1(y_1, y_2),\ F_2(y_1, y_2))$ at each of the grid points, the overall picture should show how trajectories of solutions behave in the phase plane.

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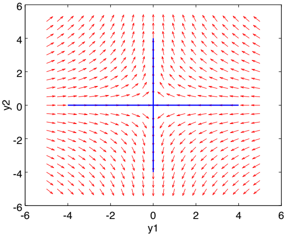
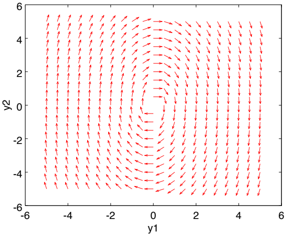
- On the following page are some direction fields around steady states at $(0, 0)$. Try to guess the *type* and *stability* of the steady states.


y2
y1
6
4
2
0
-2
-4
-6


y2
y1
6
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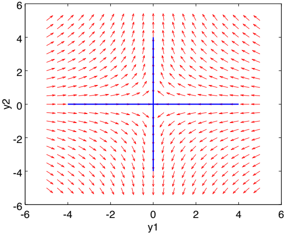
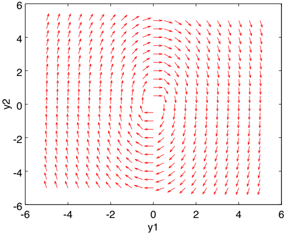
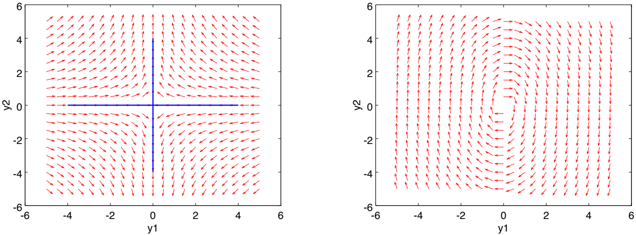
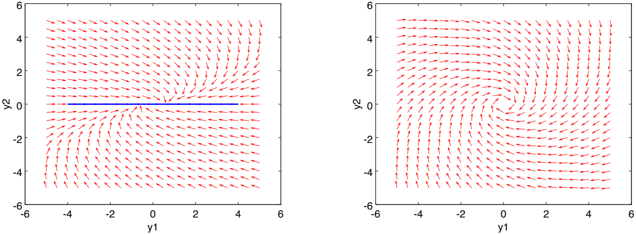


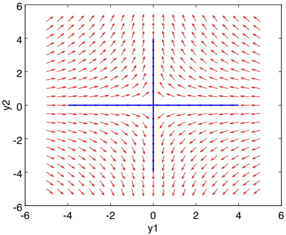


An Unstable *Saddle Point*



An Unstable *Saddle Point* **A Stable** *Centre*





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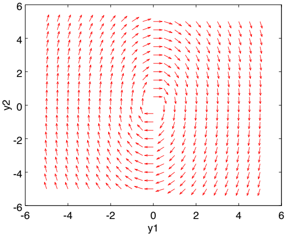
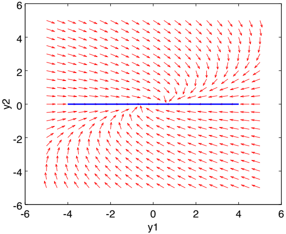
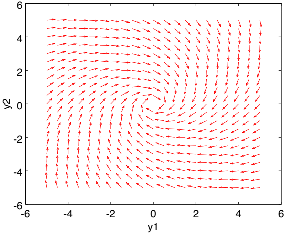
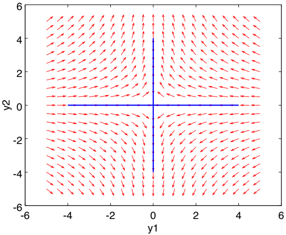


A Stable *Centre*



An Asymptotically Stable *Improper Node*





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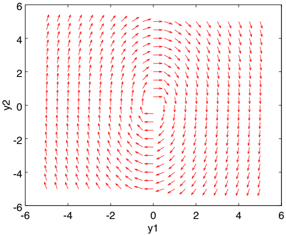

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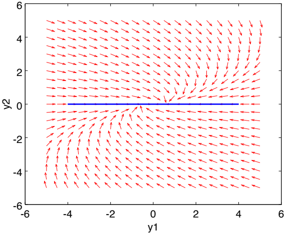

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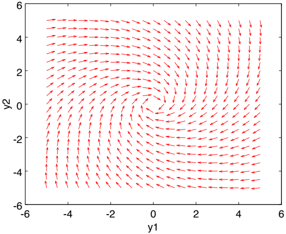

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How to Sketch Phase Portraits

↪ As we did for single ODEs, we can use calculus to help us sketch what typical solutions look like (in the phase plane) without solving a system of ODEs.

- DEFINITION In a system of n autonomous first order ODEs, $\frac{d\vec{y}}{dt} = \vec{f}(\vec{y})$, the j^{th} nullcline is the geometric shape for which

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- The following summary of how to sketch phase portraits is a slightly modified version of what is in A Primer on Mathematical Models in Biology by Segel and Edelstein-Keshet.

How to Sketch Phase Portraits - continued

Here is a systematic way of sketching trajectories in the phase plane for $\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} F_1(y_1, y_2) \\ F_2(y_1, y_2) \end{pmatrix}$ *without solving the system of ODEs*

1. If possible, find the steady states by solving the system of algebraic equations $dy_1/dt = F_1(y_1, y_2) = 0$, $dy_2/dt = F_2(y_1, y_2) = 0$. Otherwise, go to step 2.

How to Sketch Phase Portraits - continued

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$$\begin{pmatrix} \frac{dy_1}{dt} \\ \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} F_1(y_1, y_2) \\ F_2(y_1, y_2) \end{pmatrix}$$ *without solving the system of ODEs*

1. If possible, find the steady states by solving the system of algebraic equations $dy_1/dt = F_1(y_1, y_2) = 0$, $dy_2/dt = F_2(y_1, y_2) = 0$. Otherwise, go to step 2.
2. Plot the **vertical nullcline(s)**, $dy_1/dt = F_1(y_1, y_2) = 0$ and put vertical trajectories along it (them).

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   Put left-pointing arrows where $dy_1/dt < 0$, right-pointing arrows where $dy_1/dt > 0$, downward pointing arrows where $dy_2/dt < 0$, and upward-pointing arrows where $dy_2/dt > 0$.

How to Sketch Phase Portraits - continued

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We next give an example of how to construct a phase portrait (taken largely from section 7.6.1 of `A Primer on Mathematical Models in Biology` by Segel and Edelstein-Keshet).

- $\boxed{\text{EXAMPLE 23}}$: A dimensionless model for macrophage cells $m(t)$ removing dead cells $a(t)$ and killing other cells is given by the system:

$$\frac{dm}{dt} = \alpha(1-m)a - \delta m, \qquad \frac{da}{dt} = m - \eta ma - a.$$

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So clearly $\boxed{m = 0, \ a = 0}$ is a steady state.

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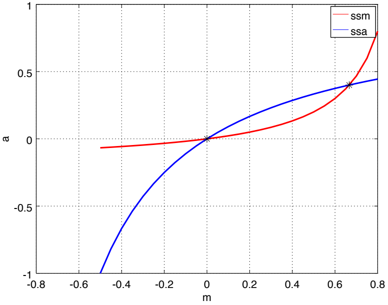
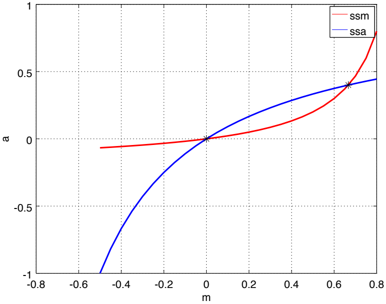
- We can now generate a graph of the nullclines and steady states using $\alpha = 1, \delta = 0.2, \eta = 1$. *(Note values outside of the first quadrant make no sense for this problem but I include them to help determine the type of steady state at the origin).*

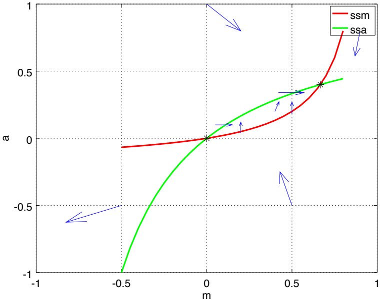


- We will next put vertical arrows across the red curve $dm/dt = 0$ and horizontal arrows across the blue curve $da/dt = 0$.

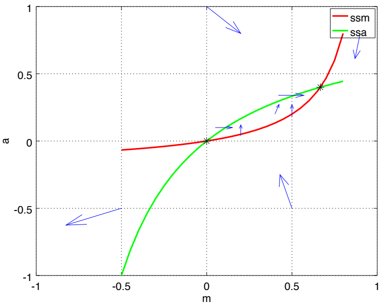


- We will next put vertical arrows across the red curve $dm/dt = 0$ and horizontal arrows across the blue curve $da/dt = 0$.
- We also calculate $dm/dt = \alpha(1-m)a - \delta m$ and $da/dt = m - \eta ma - a$ in various regions (done in Matlab) and insert appropriately-scaled vectors parallel to the $(dm/dt,\ da/dt)$ vectors and emanating from those points.

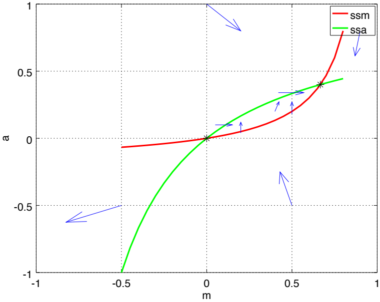
Sample point	$(dm/dt,\ da/dt)$
$(0,1)$	$(1,-1)$
$(-0.5,-0.5)$	$(-0.65,-0.25)$
$(0.5,-0.5)$	$(-0.35,1.25)$
$(0.4,0.2)$	$(0.04,0.12)$
$(0.9,0.8)$	$(-0.1,-0.62)$



- From this we see some trajectories going towards the origin and some moving away, so it is likely an *saddle point* (hence unstable).



- From this we see some trajectories going towards the origin and some moving away, so it is likely an *saddle point* (hence unstable).
- On the other hand, trajectories seem to move towards the steady state in the first quadrant, so it appears asymptotically stable and the pattern of approach suggest it is likely a node (although there are other possibilities).



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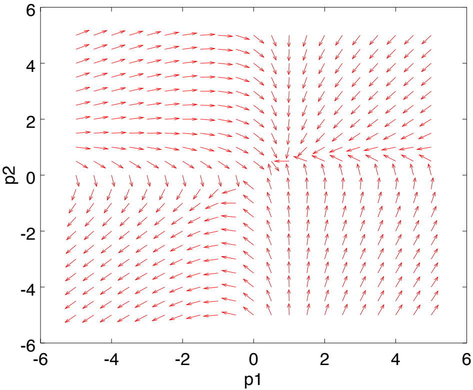

- This direction field plot confirms our conclusions on the preceding page.

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  - NOTE there are tests which do not require the calculation of the eigenvalues and just require looking at certain combinations of the entries of the matrix A *(I won't cover those in any detail but you can find them summarised in many standard ODE or Mathematical Biology books, including several on the course's reading list)*.

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- We will discuss this classification based on the eigenvalue types of A, looking at 5 cases and then summarising at the end.

$\mapsto$ First just a few general observations regarding the **EIGENVECTORS** of A, given that the solution of the ODE system is typically of the form $\vec{y} = A\vec{v}_1 e^{\lambda_1 t} + B\vec{v}_2 e^{\lambda_2 t}$.

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- Solution trajectories which do not start off on an eigenvector are generally curved and tend towards the eigenvector associated with the largest eigenvalue as t increases.

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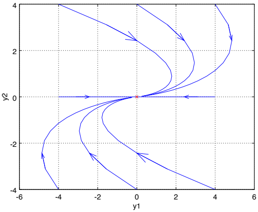
↪ This type of steady state is called a **Node**.



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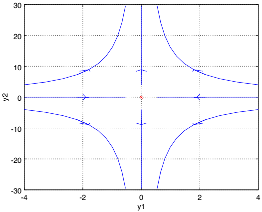

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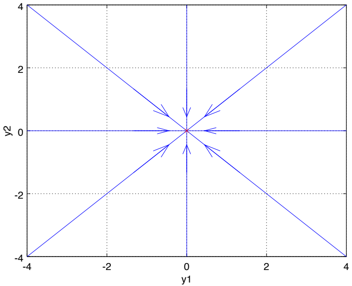
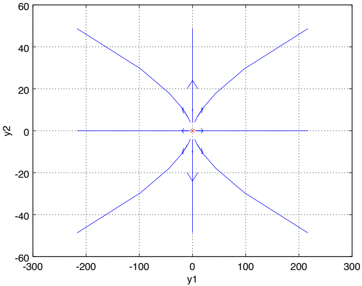
Thus solutions approach 0 as $t \to \infty$, therefore the steady state is asymptotically stable.

Proper Node



Improper Node



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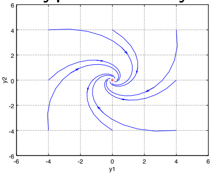
- This type of steady state is called a **Spiral Point** (or focus).



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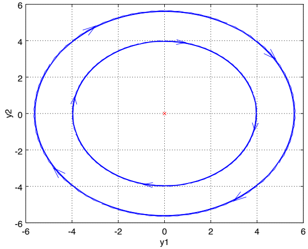

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- One nice thing that the topology of $\mathbb{R}^2$ allows is that local behaviour (near to steady states) in the phase plane can be generalised to global behaviour and one can get a good idea of how solutions behave everywhere in the plane. **This is not the case in higher dimensions**.

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In the last, ambiguous case, check the nonlinear terms or use a direction field etc. to confirm the type and stability of the steady state.

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$$\begin{pmatrix} \frac{\partial}{\partial m}[\alpha(1-m)a - \delta m] & \frac{\partial}{\partial a}[\alpha(1-m)a - \delta m] \\ \frac{\partial}{\partial m}[m - \eta ma - a] & \frac{\partial}{\partial a}[m - \eta ma - a] \end{pmatrix} = \begin{pmatrix} -\alpha a - \delta & \alpha(1-m) \\ 1-\eta a & -\eta m - 1 \end{pmatrix}$$

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$$\frac{dm}{dt} = \alpha(1-m)a - \delta m, \qquad \frac{da}{dt} = m - \eta ma - a.$$

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- Using Matlab's `eig()` function on this matrix, we see that the eigenvalues are $\mathbf{-1.67703}$ and $\mathbf{0.47703}$, so we conclude that $(0,0)$ is an (unstable) saddle point.

- Similarly at the steady state

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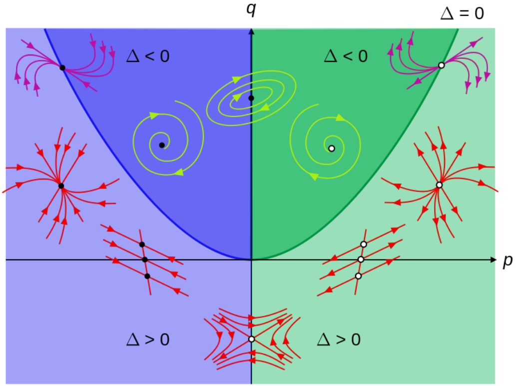
APPENDIX A

Here is an image summarising one way to categorise the steady states of a linear constant coefficient 2×2 system of ODEs without explicitly computing eigenvalues:



$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$
$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$
$$\Delta = p^2 - 4q$$

APPENDIX B - Inhomogeneous Systems $\vec{x}\,' = A\vec{x} + \vec{g}(t)$, $A_{n\times n}$

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2. The Method of Undetermined Coefficients for systems.

1. Diagonalisation Approach to Solving $\vec{x}\,' = A\vec{x} + \vec{g}(t)$

- First - review how to solve a first order linear ODE using an **integrating factor** from **Lecture 2** since it will be needed in what follows.

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- *NOTE recall that when using this diagonalisation approach with homogeneous linear systems of ODEs, we do not need to know P^{-1}. However, we DO need to know P^{-1} when solving the inhomogeneous system $\vec{x}\,' = A\vec{x} + \vec{g}(t)$ - in order to compute $P^{-1}\vec{g}(t)$.*

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$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4/3 & -1 \\ -2/3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2}{3}e^t - 1 \\ -\frac{1}{3}e^t + 1 \end{bmatrix}.$$

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$$\boldsymbol{y_1(t) = (P^{-1}\vec{x})_1 = \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t}}, \quad \textbf{where } \boldsymbol{C_1} \textbf{ is an arbitrary constant.}$$

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If, for simplicity, we let $\vec{y} = P^{-1}\vec{x}$, the first row says
$y_1' = \frac{1}{3}y_1 + \frac{1}{15}e^t + \frac{1}{5} \Rightarrow (e^{-\frac{1}{3}t}y_1(t))' = \frac{1}{15}e^{\frac{2}{3}t} + \frac{1}{5}e^{-\frac{1}{3}t} \quad \Rightarrow$

$$\boldsymbol{y_1(t) = (P^{-1}\vec{x})_1 = \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t},} \quad \textbf{where } \boldsymbol{C_1} \textbf{ is an arbitrary constant.}$$

And the second row says $y_2' = 2y_2 + \frac{1}{5}e^t - \frac{2}{5} \Rightarrow (e^{-2t}y_2)' = \frac{1}{5}e^{-t} - \frac{2}{5}e^{-2t} \Rightarrow$

$\hookrightarrow$ $\boxed{\text{EXAMPLE 15}}$ Returning to EXAMPLE 2/9, we now solve the full inhomogeneous system :

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4/3 & -1 \\ -2/3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2}{3}e^t - 1 \\ -\frac{1}{3}e^t + 1 \end{bmatrix}.$$

We saw in EXAMPLE 9 that

$$P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 2/5 & 3/5 \\ 1/5 & -1/5 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}.$$

$$\text{So we solve} \quad (P^{-1}\vec{x})' = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}(P^{-1}\vec{x}) + \begin{bmatrix} 2/5 & 3/5 \\ 1/5 & -1/5 \end{bmatrix}\begin{bmatrix} \frac{2}{3}e^t - 1 \\ -\frac{1}{3}e^t + 1 \end{bmatrix} \Rightarrow$$

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$$\boldsymbol{y_2(t) = (P^{-1}\vec{x})_2 = -\frac{1}{5}e^t + \frac{1}{5} + C_2 e^{2t},} \quad \text{where } \boldsymbol{C_2} \text{ is an arbitrary constant.}$$

$\hookrightarrow$ So $P^{-1}\vec{x} = \begin{bmatrix} \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t} \\ -\frac{1}{5}e^t + \frac{1}{5} + C_2 e^{2t} \end{bmatrix}$ so that

$$\vec{x} = P(P^{-1}\vec{x}) = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t} \\ -\frac{1}{5}e^t + \frac{1}{5} + C_2 e^{2t} \end{bmatrix} =$$

$\hookrightarrow$ So $P^{-1}\vec{x} = \begin{bmatrix} \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t} \\ -\frac{1}{5}e^t + \frac{1}{5} + C_2 e^{2t} \end{bmatrix}$ so that

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$$\begin{bmatrix} \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t} - \frac{3}{5}e^t + \frac{3}{5} + 3C_2 e^{2t} \\ \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t} + \frac{2}{5}e^t - \frac{2}{5} - 3C_2 e^{2t} \end{bmatrix} \Rightarrow$$

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$$\boldsymbol{\vec{x}(t) = \begin{bmatrix} -\frac{1}{2}e^t + C_1 e^{\frac{1}{3}t} + 3C_2 e^{2t} \\ \frac{1}{2}e^t - 1 + C_1 e^{\frac{1}{3}t} - 2C_2 e^{2t} \end{bmatrix}}$$

$\hookrightarrow$ So $P^{-1}\vec{x} = \begin{bmatrix} \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t} \\ -\frac{1}{5}e^t + \frac{1}{5} + C_2 e^{2t} \end{bmatrix}$ so that

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$$\boldsymbol{\vec{x}(t) = \begin{bmatrix} -\frac{1}{2}e^t + C_1 e^{\frac{1}{3}t} + 3C_2 e^{2t} \\ \frac{1}{2}e^t - 1 + C_1 e^{\frac{1}{3}t} - 2C_2 e^{2t} \end{bmatrix}}$$

$\hookrightarrow$ We now use the initial conditions $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to get

$$\begin{aligned} C_1 + 3C_2 &= 3/2 \\ C_1 - 2C_2 &= 3/2 \end{aligned}$$

$\hookrightarrow$ So $P^{-1}\vec{x} = \begin{bmatrix} \frac{1}{10}e^t - \frac{3}{5} + C_1 e^{\frac{1}{3}t} \\ -\frac{1}{5}e^t + \frac{1}{5} + C_2 e^{2t} \end{bmatrix}$ so that

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$$\boldsymbol{\vec{x}(t) = \begin{bmatrix} -\frac{1}{2}e^t + C_1 e^{\frac{1}{3}t} + 3C_2 e^{2t} \\ \frac{1}{2}e^t - 1 + C_1 e^{\frac{1}{3}t} - 2C_2 e^{2t} \end{bmatrix}}$$

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$$\boldsymbol{\vec{x}(t) = \begin{bmatrix} -\frac{1}{2}e^t + C_1 e^{\frac{1}{3}t} + 3C_2 e^{2t} \\ \frac{1}{2}e^t - 1 + C_1 e^{\frac{1}{3}t} - 2C_2 e^{2t} \end{bmatrix}}$$

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$$\begin{aligned} C_1 + 3C_2 &= 3/2 \\ C_1 - 2C_2 &= 3/2 \end{aligned}$$

$\Rightarrow C_2 = 0, \quad C_1 = \frac{3}{2}$. So the solution to the initial value problem is $\boldsymbol{\vec{x}(t) = \begin{bmatrix} -\frac{1}{2}e^t + \frac{3}{2}e^{\frac{1}{3}t} \\ -1 + \frac{1}{2}e^t + \frac{3}{2}e^{\frac{1}{3}t} \end{bmatrix}}$, as expected.

2. Method of Undetermined Coefficients Approach to Solving $\vec{x}\,' = A\vec{x} + \vec{g}(t)$

(If pressed for time, you can ignore this is you have not seen the Method of Undetermined Coefficients before for solving constant coefficient linear 2nd order inhomogeneous ODEs)

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↪ There isn't much new here if you have seen the Method of Undetermined Coefficients for second order constant coefficient linear ODEs. Basically, we can find a **particular solution** to $\boldsymbol{\vec{x}\,' = A\vec{x} + \vec{g}(t)}$ *in the special case where A is constant and $\vec{g}(t)$ contains sines, cosines, polynomials, exponential functions, or sums/products of these.*

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- ↪ Again assume a solution $\vec{x}_p(t)$ of the form of the various entries of $\vec{g}(t)$ with undetermined coefficients, substitute this assumption for $\vec{x}_p(t)$ into $\boldsymbol{\vec{x}\,' = A\vec{x} + \vec{g}(t)}$, and find out the values of those coefficients.

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- Once we have a **particular solution** $\vec{x}_p(t)$ to $\boldsymbol{\vec{x}\,' = A\vec{x} + \vec{g}(t)}$ and also know the *general solution of the HOMOGENEOUS system* $\boldsymbol{\vec{x}\,' = A\vec{x}}$ (the **complementary function**) $\vec{x}_c(t)$, then the GENERAL SOLUTION of $\boldsymbol{\vec{x}\,' = A\vec{x} + \vec{g}(t)}$ is simply

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- ▶ *NOTE that this is identical to how we use the Method of Undetermined coefficients to solve linear single ODEs such as $ax'' + bx' + cx = g(t)$.*

$\hookrightarrow$ $\boxed{\text{EXAMPLE 16}}$ We will return to EXAMPLEs 9 and 15 to solve $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4/3 & -1 \\ -2/3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2}{3}e^t - 1 \\ -\frac{1}{3}e^t + 1 \end{bmatrix}$, *BUT now using the Method of Undetermined Coefficients.*

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We already know, from EXAMPLE 9, the general solution of the homogeneous part of this equation:

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Now we find a **particular integral** by observing $\vec{g}(t) = \begin{bmatrix} \frac{2}{3}e^t - 1 \\ -\frac{1}{3}e^t + 1 \end{bmatrix}$ and assuming

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$\vec{x}_p\,'(t) = \begin{bmatrix} ae^t \\ ce^t \end{bmatrix}$, and substituting this into $\boldsymbol{\vec{x}\,' = A\vec{x} + \vec{g}(t)}$, we get

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$$\begin{bmatrix} ae^t \\ ce^t \end{bmatrix} = \begin{bmatrix} 4/3 & -1 \\ -2/3 & 1 \end{bmatrix}\begin{bmatrix} ae^t + b \\ ce^t + d \end{bmatrix} + \begin{bmatrix} \frac{2}{3}e^t - 1 \\ -\frac{1}{3}e^t + 1 \end{bmatrix} =$$

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$$\begin{bmatrix} ae^t \\ ce^t \end{bmatrix} = \begin{bmatrix} 4/3 & -1 \\ -2/3 & 1 \end{bmatrix}\begin{bmatrix} ae^t + b \\ ce^t + d \end{bmatrix} + \begin{bmatrix} \frac{2}{3}e^t - 1 \\ -\frac{1}{3}e^t + 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{4}{3}ae^t + \frac{4}{3}b - ce^t - d \\ -\frac{2}{3}ae^t - \frac{2}{3}b + ce^t + d \end{bmatrix} + \begin{bmatrix} \frac{2}{3}e^t - 1 \\ -\frac{1}{3}e^t + 1 \end{bmatrix}$$

REMINDER: $\vec{x}_c(t) = B_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{\frac{1}{3}t} + B_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{2t}$ and $\vec{x}_p(t) = \begin{bmatrix} ae^t + b \\ ce^t + d \end{bmatrix}$

This simplifies to $$\begin{bmatrix} ae^t \\ ce^t \end{bmatrix} = \begin{bmatrix} (\frac{2}{3} + \frac{4}{3}a - c)e^t + \frac{4}{3}b - d - 1 \\ (-\frac{2}{3}a + c - \frac{1}{3})e^t - \frac{2}{3}b + d + 1 \end{bmatrix}.$$

REMINDER: $\vec{x}_c(t) = B_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{\frac{1}{3}t} + B_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{2t}$ and $\vec{x}_p(t) = \begin{bmatrix} ae^t + b \\ ce^t + d \end{bmatrix}$

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Equating the coefficients of like terms on either side of the equations (and simplifying), we conclude that

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$\hookrightarrow$ $\boxed{\text{FINAL NOTE:}}$ When solving $\vec{\boldsymbol{x}}\,' = \boldsymbol{A}\vec{\boldsymbol{x}} + \vec{\boldsymbol{g}}(\boldsymbol{t})$ using the Method of Undetermined Coefficients, there is only one case in which the approach differs slightly from that used in the solving of equations like $ax'' + bx' + cx = g(t)$.

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$\rightsquigarrow$ *If the initial assumed form of the particular solution $\vec{x}_p(t) = \vec{a}e^{\lambda t}$, where λ is an eigenvalue of A (so that the term $\vec{a}e^{\lambda t}$ already appears in the complementary function), then instead of adjusting the assumption to $\vec{x}_p(t) = t\vec{a}e^{\lambda t}$, also include lower order terms in the assumption:* $\boxed{\vec{\boldsymbol{x}}_{\boldsymbol{p}}(\boldsymbol{t}) = \boldsymbol{t}\vec{\boldsymbol{a}}\boldsymbol{e}^{\lambda \boldsymbol{t}} + \vec{\boldsymbol{b}}\boldsymbol{e}^{\lambda \boldsymbol{t}}}$, where $\vec{a}$ and $\vec{b}$ are constant vectors whose entries are to be determined by substitution into the ODE system $\vec{\boldsymbol{x}}\,' = \boldsymbol{A}\vec{\boldsymbol{x}} + \vec{\boldsymbol{g}}(\boldsymbol{t})$.