TIPS

1. Invest time learning the *language* of Mathematics (with all of its special cases and exceptions and conventions).
2. Review constantly.
3. Do assignments, tutorials, etc. Practise, practise, practise. Read textbook/supplementary notes. Ideally read the textbook/supplementary notes material on a topic *before* the relevant lecture. You will have to do significant work outside of the classroom to master the material.
4. Attend and engage with lectures and tutorials. This represents the most efficient way to learn the material. Come prepared and ask questions if you do not understand something.
5. Speak to me as soon as you feel you may be falling behind.
6. In summary: KEEP UP. This will be a very fast-paced class and falling behind is **very unwise**. Mathematics is very hierarchical and you generally progress only by first knowing well what went before.

Mathematics for the Life Sciences - Introduction

$\hookrightarrow$ PRACTICAL INFORMATION $\hookleftarrow$

- Lecturers: **Erwin George** ($\approx$ weeks 2 - 8) and **Tony Mann** ($\approx$ week 1 and last 3 weeks) $\rightsquigarrow$ (E.George@gre.ac.uk and A.Mann@gre.ac.uk).
- No exam. A single coursework $\rightsquigarrow$ see course handbook for release and due dates.
- The main lecture notes will typically be provided in three formats (Tony will likely only use 2 formats):
  1. A printable version with parts missing. *You are expected to print this out and read it BEFORE the lecture and to try to do the examples*. **Bring this printout with you to the lecture in order to complete the notes.**
  2. A complete printable version will be made available (some time) after lectures.
  3. A complete in-class version which attempts to mimic the way the lecture was delivered in class. I recommend using this version if you miss a lecture to get caught up on that lecture.
- See also the *Course Handbook* at the top of the class Moodle page for a summary of key information relevant to the course.

$\mapsto$ **Provisional Summary of Course Content:**

1. General introduction to applications of mathematics in the life sciences $\rightsquigarrow$ 1 WEEK;
2. Review of first order single ordinary differential equations (ODEs) - (including exact solutions, geometric/qualitative approaches, numerical methods) $\rightsquigarrow$ 1 WEEK;
3. Modelling with ODEs and single species population dynamics $\rightsquigarrow$ 2 WEEKS;
4. Systems of first order ODEs (including numerical methods and qualitative approaches) $\rightsquigarrow$ 2 WEEKS;
5. Population dynamics of interacting species $\rightsquigarrow$ 2 WEEKS.
6. Infectious disease $\rightsquigarrow$ 1 WEEK;
7. Population genetics and evolution, and the use of Game Theory in the life sciences $\rightsquigarrow$ 3 WEEKS.

$\rightsquigarrow$ *This simple introduction will give you the tools to explore other applications of mathematics to life sciences, such as biochemical (including enzyme) kinetics, and I will point some of these out to you as we proceed:*

Suggested Reading List

- **Mathematics of Life: Unlocking the Secrets of Existence** by Ian Stewart *(Profile)*.
- **Essential Mathematical Biology** by Nicholas Britton *(Springer)*
- **Mathematical Biology: I. An Introduction** by J.D. Murray *(Springer)*
- **Mathematical Models in Biology** by Leah Edelstein-Keshet *(SIAM)*
- **A Primer on Mathematical Models in Biology** by Lee Segel and Leah Edelstein-Keshet *(SIAM)*
- **Super Cooperators: Evolution, Altruism and Human Behaviour or Why We Need Each Other to Succeed** by Martin Nowak and Roger Highfield *(Canongate)*
- **When maths doesn't work: what we learn from the Prisoners' Dilemma** by Tony Mann - *Lecture transcript and video available at http://www.gresham.ac.uk/lectures-andevents/when-mathsdoesnt-work-what-welearn-from-theprisoners-dilemma;*

INTRODUCTION

- Many *(but not all!)* of the mathematical models of life science applications we will encounter in this course will be Ordinary Differential Equations (ODEs) or systems of ODEs, typically with *initial conditions*, so that we have an Initial Value Problems (IVPs).
- Therefore, before beginning to build and study these models, we will review relevant aspects of ODEs.
- NOTE there are other key mathematical tools for modelling life science problems - such as stochastic differential equations, partial differential equations and boundary value problems, etc. - which we will not cover in this course.
- **As always, NOTATION and LANGUAGE will be very important in what follows; pay close attention to it and ensure that you learn and understand the notation and language used!**

Introduction continued

- A *mathematical model* of a real-world problem often takes the form of a **differential equation** (rates of change, slopes, deflections $\rightsquigarrow$ derivatives of a function).

- **Differential Equation** (DE) $=$ an equation that contains an *unknown function* and some of its derivatives. The objective is typically to find out what that unknown function is. *Obviously, this is much trickier than just solving an algebraic equation for a single variable or a finite set of variables*. However, we are also often just interested in the *qualitative* behaviour of the solution(s) - such as its (their) asymptotic behaviour as the independent variable time $\to \infty$.

- Differential equations in which the unknown function is a function of only **one** independent variable are called Ordinary Differential Equations (ODEs), whereas differential equations involving an unknown *multivariable function* are called Partial Differential Equations (PDEs).
  We will essentially only study ODEs in this course $\rightsquigarrow$ *you can choose to study PDEs in the course Numerical Solution of PDEs which runs concurrently*.

- These ODE models may also consist of n different ODEs involving n unknown functions and their derivatives.
  We will also study systems of ODEs in this course.

- $\rightsquigarrow$ Example $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^x$ $\rightsquigarrow$ a **second order** differential equation since the highest order derivative involved is 2. What is the independent variable in this equation? .

- $\rightsquigarrow$ Example $y' = f(x, y)$ represents a **general first order differential equation**. $f(x, y)$ $\rightsquigarrow$ *a function of the two independent variables* x *and* y*, e.g.* $f(x, y) = x^2y + \sin(x - y)$. What is the independent variable in this DE? .

- $\rightsquigarrow$

$$\sin(t)\frac{dx}{dt} - 4\frac{dy}{dt} = \cos(t)$$

$$5\frac{dx}{dt} + e^{2t}\frac{dy}{dt} = 5$$

is an example of system of 2 first order ODEs. The independent variable is and the unknown functions (dependent variables) are .
(Warning - the terminology used in some books can be a bit confusing, referring to this as a "**second** order system" since it contains **two** equations. I prefer the more explicit description "a system of 2 first order equations").

- $\hookrightarrow$ Solving a first order DE $\rightsquigarrow$ integration (hence one constant of integration) in the solution (function).
- $\hookrightarrow$ Solving a second order DE $\rightsquigarrow$ integration **twice** (hence *two* constants of integration in the solution function) ... etc.
- $\mapsto$ The solution which contains the constants of integration usually describes all possible solutions and is called the General Solution (GS) of the DE.
- $\mapsto$ Usually, the scenario the ODE models has extra pieces of information – called Initial Conditions *(or boundary conditions)* – which can be used to determine the constant of integration and hence get a unique (function) solution to the DE, called a Particular Solution (PS). Clearly, the number of *initial conditions* must match the order of the DE *(= the number of constants of integration that "solving" the equation produces)*.
- **Key Definition** **Initial Value Problem (IVP)** = `a DE with one or more initial conditions.`
  We will focus on Initial Value Problems and not Boundary Value Problems (where boundary conditions instead of initial conditions are specificied) in this course.
- $\mapsto$ Example of **initial condition** for $\frac{dx}{dt} = g(x, t)$: $x = b$ at $t = a$, also written $x(a) = b$ (where a and b are constants).
- **I will often just use the generic terms ODE or system of ODEs to also include the possibility of initial conditions being present - so to include IVPs or systems of IVPs.**

3 Types of DEs We'll Learn to Solve Now

1. *Integration in disguise equations* $\rightsquigarrow$ these will be differential equations of *ANY* order *(i.e., where the highest order derivative of the unknown function can be first or second or third or fourth or fifth, ...)*, in which solving the differential equation just involves possibly rearranging the equation then integrating both sides one or more times - using the Fundamental Theorem of Calculus.
2. *Separable* **first order** *equations* $\rightsquigarrow$ several of the key equations we encounter will fall into this category.
3. *Linear* **first order** *equations*.

$\mapsto$ **It is important that you learn how to recognise which of the three categories a given differential equation falls into so that you will know exactly how to solve it** *(or whether you can solve it)***!**

Integration in Disguise Differential Equations

- The first class of ODEs we will solve are those which **can be** written in the form

$$\frac{dy}{dx} = f(x) \ \text{ or } \ \frac{d^2y}{dx^2} = f(x) \ \text{ or } \ \frac{d^3y}{dx^3} = f(x), \ \ldots, \ \text{etc.}$$

  i.e. equations which can be written in the form

$$\text{Some derivative of the unknown function } =$$

$$\text{a function of the independent variable only } (\textit{including CONSTANTS}).$$

- By the **Fundamental Theorem of Calculus**, such equations can be solved by integrating the right hand side function (n times for a DE of order n) with respect to the independent variable.

  $\rightsquigarrow$ For example $\frac{dy}{dx} = f(x) \Rightarrow \int \frac{dy}{dx}\, dx = \int f(x)\, dx$ OR $\boldsymbol{y = \int f(x)\, dx}$.

- $\boxed{\text{EXAMPLE 1}}$ Find the general solution of

$$\frac{d^3y}{dx^3} - \sin(x) = 4x^3.$$

⋆ $\boxed{\text{ANSWER}}$ First rewrite as

$$\frac{d^3y}{dx^3} = \sin(x) + 4x^3.$$

Now integrate three times to get the answer:

$$\frac{d^2y}{dx^2} = -\cos(x) + x^4 + C_1$$

$$\Rightarrow \frac{dy}{dx} = -\sin(x) + \frac{1}{5}x^5 + C_1x + C_2$$

$$\boldsymbol{\Rightarrow y(x) = \cos(x) + \frac{1}{30}x^6 + \frac{C_1}{2}x^2 + C_2x + C_3.}$$

CHECK YOUR ANSWER!!!

$\hookrightarrow$ EXAMPLE 2 Find the general solution for $e^y\left(5 - \frac{dz}{dy}\right) = 6 - 10\cosh y$.

$\mapsto$ ANSWER What is the independent variable? .

$\mapsto$ What is the particular solution if $z(0) = 3$?

.

$\hookrightarrow$ EXAMPLE 3 - *Introductory Modelling (simple mechanics): for this, you need to know that the derivative of displacement/position with respect to time is velocity, and the derivative of velocity with respect to time is acceleration*:
The acceleration of a bus along a straight road is given by $a(t) = 3t$. We start observing the bus ($t = 0$) when it is 50 m along the road, and 1 s after we begin observing it, its velocity is 5 m/s. Write this as an ODE with "initial" conditions, and find a particular solution for the *position* of the bus along the road at time t.

$\mapsto$ ANSWER Letting $x(t)$ be the position of the bus t seconds after we begin observing it, then

$$x(t) = \frac{1}{2}t^3 + \frac{7}{2}t + 50.$$

Separable Equations

- Definition: A separable equation (for the function $y(x)$) is a **first order** DE which **can be** written in the form

$$\frac{dy}{dx} = f(x)g(y). \tag{1}$$

  i.e., the DE is separable if $\frac{dy}{dx}$ can be written as the product (or quotient) of a function of x *alone* **and** a function of y *alone*.

- To solve Equation (1), rearrange it into the form

$$\frac{1}{g(y)}\,dy = f(x)\,dx$$

  and then integrate both sides, *i.e.*

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx. \tag{2}$$

- NOTE after integration, only **ONE** constant of integration is needed in Equation (2), and it is not always feasible to solve the resulting equation for y as function of x.

REMINDER: $\frac{dy}{dx} = f(x)g(y) \rightsquigarrow \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

$\hookrightarrow$ For those who are worried that we are integrating with respect to *different* variables on both sides of the equation, here is why it works *(from the substitution rule ... this proof is also available in most calculus textbooks)*:

$$\int \frac{1}{g(y)} dy = \int \frac{1}{g(y(x))} \frac{dy}{dx} dx$$
$$= \int \frac{1}{g(y(x))} f(x) g(y(x)) dx = \int f(x) dx.$$

- **NOTE** this method of solution is sometimes referred to as **separation of variables**.

$\hookrightarrow$ EXAMPLE 4 Solve $\frac{du}{dt} = e^{u+3t}$.

$\mapsto$ ANSWER

$$u(t) = -\ln\left(K - \frac{1}{3}e^{3t}\right).$$

- Of course, it isn't always possible or easy to solve for the dependent variable (u in this case) in terms of the independent variable (t).
- In this case, we can solve for u explicitly so **we can check directly that the solution satisfies the DE**:

$\hookrightarrow$ $\boxed{\text{EXAMPLE 5}}$ Solve $y' = \frac{y \cos x}{1+2y^2}, \quad y(0) = 1.$

$\mapsto$ $\boxed{\text{ANSWER}}$

$$(4)$$

Linear First Order Equations

- **Definition:** A linear first order ODE is on which *can be written as*

$$(\textit{Derivative of unknown function}) + (\textit{function of independent variable}) \times (\textit{unknown function})$$

$$= \textit{function of independent variable}$$

- To solve the general first order *linear* ODE, $\boxed{\frac{dy}{dt} + p(t)y = r(t)}$, we seek to multiply both sides of the equation by an *integrating factor* $\mathrm{I}(t)$ chosen so that the left hand side of the equation becomes, by the **product rule (in reverse)**, $\boldsymbol{\frac{d}{dt}\left[\mathrm{I}(t)y\right]}$. NOTE this would then make it easy to use the **Fundamental Theory of Calculus** to solve for $y(t)$.
  - The modified linear first order ODE would then look like

$$\mathrm{I}(t)\frac{dy}{dt} + \mathrm{I}(t)p(t)y = \frac{d}{dt}\left[\mathrm{I}(t)y\right] = I(t)r(t) \tag{6}$$

Reminder: $\mathrm{I}(t)\frac{dy}{dt} + \mathrm{I}(t)p(t)y = \frac{d}{dt}\left[\mathrm{I}(t)y\right] = I(t)r(t)$ (6)

- The obvious thing to do is to apply the product rule to differentiating $\mathrm{I}(t)y$ and hope that by matching terms/comparing with the left hand side of the modified linear first order ODE, Equation (6), we can figure out what $\mathrm{I}(t)$ should be. So $\frac{d}{dt}\left[\mathrm{I}(t)y\right] = \frac{d\mathrm{I}}{dt}y + \mathrm{I}\frac{dy}{dt}$ by the product rule. Comparing that to the left hand side of Equation (6), $\mathrm{I}(t)\frac{dy}{dt} + \mathrm{I}(t)p(t)y$, we conclude that $\mathrm{I}(t)p(t)$ **must** $= \frac{d\mathrm{I}}{dt}$.
- That last equation can be viewed as a *separable* ODE: $\frac{d\mathrm{I}}{dt} = \mathrm{I}p(t) \Rightarrow$

  $$\frac{1}{\mathrm{I}}\, d\mathrm{I} = p(t)\, dt \Rightarrow \int \frac{1}{\mathrm{I}}\, d\mathrm{I} = \int p(t)\, dt.$$

  Ignoring constants of integration (we only need *one* function $\mathrm{I}(t)$ to serve as the integrating factor), we solve to get **ln** $\boldsymbol{[\mathrm{I}(t)] = \int p(t)\, dt \Rightarrow \mathrm{I}(t) = e^{\int p(t)\, dt}}$.
- Hence, the integrating factor to choose when solving $\frac{dy}{dt} + p(t)y = r(t)$ is **always**

  $$\mathrm{I}(t) = e^{\int p(t)\, dt}. \tag{7}$$

Reminder: $\mathrm{I}(t)\frac{dy}{dt} + \mathrm{I}(t)p(t)y = \frac{d}{dt}\left[\mathrm{I}(t)y\right] = I(t)r(t)$ (6)

- The ODE, Equation (6), then becomes $\frac{d}{dt}\left(e^{\int p(t)\,dt}y\right) = e^{\int p(t)\,dt}r(t)$, and by the Fundamental Theorem of Calculus we integrate both sides with respect to t to get

$$e^{\int p(t)\,dt}y = \int \left[e^{\int p(t)\,dt}r(xt\right]\,dx + C, \tag{8}$$

from which it is easy to find y:

$$\boxed{y = e^{-\int p(x)\,dx}\left\{\int \left[e^{\int p(x)\,dx}r(x)\right]\,dx\right\} + Ce^{-\int p(x)\,dx}}.$$

$\hookrightarrow$ Instead of focussing on the above formula, I recommend remembering the *process* of multiplying the equation by an integrating factor to get the product rule in reverse, then using the **Fundamental Theorem of Calculus** to solve the resulting modified ODE.

- $\boxed{\text{EXAMPLE 6}}$ Find the General Solution of $2t\frac{dy}{dt} + y = 3t$.
- $\boxed{\text{ANSWER}}$ First, write the equation in standard form, $\frac{dy}{dt} + \frac{1}{2t}y = \frac{3}{2}$.

  Next, the *integrating factor* is

$$So \quad y(t) = t + Ct^{-\frac{1}{2}} \quad (CHECK!).$$

- EXAMPLE 7

  (a) Solve the initial value problem $t\frac{dy}{dt} = 1 + t^3, \quad y(1) = \frac{4}{3}$.

  (b) Find the general solution of $\frac{dx}{dt} = 4x + 2e^{5t}$.

  (c) Solve the initial value problem $\frac{y}{\cos(x)}\frac{dy}{dx} = 1, \quad y(0) = 4$.

NOTE it is possible for a differential equation to fall into more than one of the categories *integration in disguise*, *separable*, or *linear* - in which case you can select the solution technique you prefer.

- For example, the equation in part (a) of **EXAMPLE 7** could also have been viewed as a linear ODE (in addition to *integration in disguise)*. We will re-do the problem treating the ODE as linear: Solve the initial value problem $t\frac{dy}{dt} = 1 + t^3, \quad y(1) = \frac{4}{3}$.

- $\boxed{\text{ANSWER}}$ In standard form this equation is $\frac{dy}{dt} + 0y = \frac{1}{t} + t^2$. So the integrating factor is $e^{\int 0\ dt} = e^0 = 1$ and thus

$$\frac{d}{dt}(1\times y) = 1\times\left(\frac{1}{t} + t^2\right) \Rightarrow 1\times y = \int \left(\frac{1}{t} + t^2\right)\ dt = \ln t + \frac{t^3}{3} + C.$$

And as in EXAMPLE 7, the initial condition means $C = 1$.

- Can you think of a first order ODE which could be considered both *linear* and *separable*? See **Tutorial 2** for some more cases of ODEs which fall into more than one of the three categories we have considered.

- **KEY DEFINITION**: `An ODE which is not linear is called` **nonlinear**.
- NOTE for now I have only officially defined what it means for a *first* order ODE to be linear, however the definition generalises in the "natural" way for higher order ODEs.

  For example, A linear *second* order ODE for $y(t)$ is on which *can be written as*

$$\frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_2(t)y = r(t).$$

- For example, the separable ODEs in EXAMPLE 4 ($\frac{du}{dt} = e^{u+3t}$) and EXAMPLE 5 ($y' = \frac{y\cos x}{1+2y^2}$) are *nonlinear* since they cannot be written in the form $\frac{dy}{dt} + p(t)y = r(t)$.

- Are the ODEs from EXAMPLE 1 $\frac{d^3y}{dx^3} - \sin(x) = 4x^3$, EXAMPLE 2 $e^y\left(5 - \frac{dz}{dy}\right) = 6 - 10\cosh y$), and EXAMPLE 3 $x''(t) = 3t$ linear or nonlinear?
  - ANSWER

- Linear DEs have many nice properties which nonlinear DEs do not have. For example, if $y_1(t)$ and $y_2(t)$ are two solutions to a linear **homogeneous** *(right hand side function $g(t) \equiv 0$)* DE, so is $ay_1(t) + by_2(t)$ for any constants a and b **(not the case for nonlinear DEs)**.

Existence and Uniqueness of Solutions of IVPs

For the initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0 :$$

↪ If f and $\frac{\partial f}{\partial y}$ are *continuous* in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0), then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there exists a unique solution to the above IVP.

⇝ A proof of this is quite advanced and is omitted. A simplified version can be found in section 2.8 of `Elementary Differential Equations` by *Boyce and Diprima* and more thorough proofs and discussions of existence and uniqueness of solutions can be found in books such as `Ordinary Differential Equations` by *Birkhoff and Rota* (for example, in sections 10-12 of chapter 1).

- In the case of linear first order IVPs $y' + p(t)y = r(t)$, $\;y(t_0) = y_0$, the above theorem becomes: If p and g are continuous on an open interval containing the point $t = t_0$, then there exists a unique solution to the IVP on that same open interval containing t_0.

Autonomous First Order ODEs - a geometric look at solution trends

↦ **Introduction** Sometimes, the solution of an ODE (which may be hard to find) is not so important; instead, *patterns* or *general behaviours* of the solutions are good enough for the situation being modelled, *and these can be determined by a geometric analysis of the ODE without solving it!*.

- This is often the case, for example, with population models where we sometimes just want to see the long term trend - *i.e.*, what happens to the population as time $\to \infty$, or see if there are certain threshold population values on either side of which the population evolution is very different.

- **KEY DEFINITION**: If a *first order* ODE can be written in the form $\boxed{\frac{dy}{dt} = f(y),}$ so that the right-hand-side is a function of the dependent variable y only, then the ODE is called **autonomous**. Otherwise, it is called non-autonomous (or just *not autonomous*).

- For example, $\frac{dy}{dt} = y^2 - 6y - 8e^y$ is autonomous but $\frac{dy}{dt} = t^2 - 6ty$ is not.
- Many *(but not all)* of the ODEs we study qualitatively will be autonomous.

- Note autonomous ODEs are sometimes "relatively" easy to solve, using the *separation of variables* technique $\rightsquigarrow$ see, for example, **Tutorial 2**.

  However we will often be interested in qualitatively studying the trends in their solution *without actually solving them*.

$\mapsto$ In an autonomous ODE $\dfrac{dy}{dt} = f(y)$, if $f(y)$ is differentiable *(hence continuous)* then any **root(s)** of $f(y)$ is/are called equilibrium point(s) or critical point(s) of the ODE.

↦ If $y = y_0$ is an equilibrium point of $\frac{dy}{dt} = f(y)$, so

$$\boldsymbol{\frac{dy}{dt} = f(y_0) = 0}$$

then $y(t) = y_0$ is a solution to the same ODE with the special initial condition $y(t_0) = y_0$. Thus y_0 is also called an equilibrium solution of the ODE.

↦ **KEY DEFINITION** For autonomous ODE

$$\frac{dy}{dt} = f(y),$$

the (constant) solutions to $f(y) = 0$ are called equilibrium points, critical points, or equilibrium solutions to the ODE.
Thus if $f(y_0) = 0$ then $y(t) = y_0$ is an equilibrium solution of the ODE $\frac{dy}{dt} = f(y)$.

- On a graph of solutions y versus t of the ODE $\frac{dy}{dt} = f(y)$ where we are only interested in solutions over the interval $t \geq t_0$, the graphs of the equilibrium solutions would be **horizontal lines** which would separate the plane (for $t \geq t_0$) into regions where the general behaviour as $t \to \infty$ of other solutions would be the same.

$\mapsto$ **Hence, finding equilibrium solutions of $\frac{dy}{dt} = f(y)$ is a key first step to identifying the general behaviour of solutions as $t \to \infty$.**

Example:

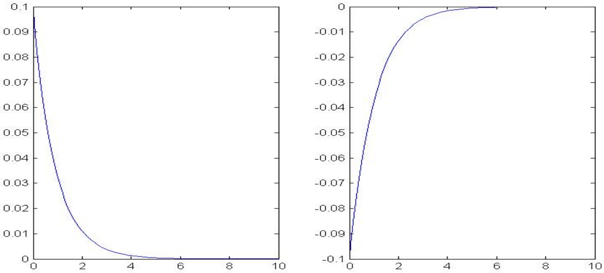
$$\frac{dy}{dt} = \frac{y}{y-1}, \qquad y \neq 1$$

clearly $y_0 = 0$ is an equilibrium point. It is not difficult to solve this equation and the solution is

$$y(t) - \ln y(t) = t + C$$

but we cannot write $y(t)$ as an explicit function of t . Despite this we know *from the equation* that there is a constant zero solution for a zero starting value.

What happens for this case if we start off with a **nearby** non-zero value e.g. 0.1 or -0.1 ? The solution simply moves to the equilibrium value i.e. this value is in some sense stable.



$\mapsto$ In the definitions which follow, when the statement *solutions which start out near to y_0* (where $y(t) = y_0$ is an equilibrium solutions of $\frac{dy}{dt} = f(y)$), means *solutions $y(t)$ of $\frac{dy}{dt} = f(y)$ for which $y(t_0) = y_0 \pm \epsilon$, where ϵ is small enough so that we do not cross over another equilibrium solution*. In other words, *solutions which start out within one of the regions determined by the horizontal lines given by all of the equilibrium solutions of $\frac{dy}{dt} = f(y)$*. Note such solutions are themselves NOT constant.

- For the autonomous differential equation $\dfrac{dy}{dt} = f(y)$ an *equilibrium solution* y_0 is:
  - **stable/attracting** if solutions $y(t)$ which start out near to y_0 move closer to y_0 as $t \to \infty$;
  - **unstable/repelling** if solutions $y(t)$ which start out near to y_0 move away from y_0 as $t \to \infty$;
  - **semistable** if some solutions $y(t)$ which start out near to y_0 move closer to y_0 and some move away from y_0 as $t \to \infty$. In general, solutions that start out below y_0 will move away from y_0 AND solutions that start out above y_0 will move towards y_0 OR vice versa.

The basic rules for classifying an equilibrium solution y_0 of $\frac{dy}{dt} = f(y)$ as *stable*, *unstable*, or *semistable* can be easily figured out using basic calculus and the aid of two related graphs to figure out how solutions starting out close to that equilibrium point behave:

1. $f(y)$ against y *(showing the graph crossing the y axis at $y = y_0$)*, and

2. $y(t)$ versus t *(showing the equilibrium solution $y(t) = y_0$)*.

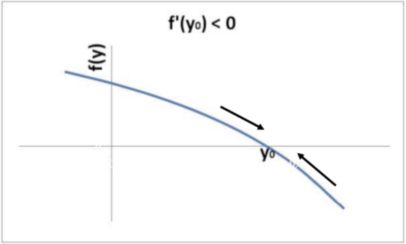
$\mapsto$ In the upcoming few slides, the images are, without loss of generality, for an equilibrium point of $y_0 = 2$.

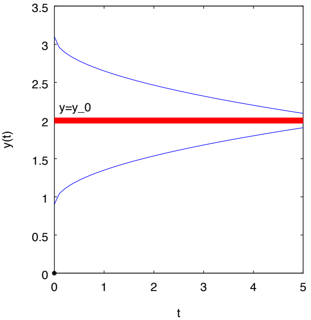
- The key to understanding the behaviour of solutions close to a critical/stationary/equilibrium point, $y = y_0$, is *to look at the gradient of the RHS at that equilibrium point - i.e.*

$$\left.\frac{df(y)}{dy}\right|_{y=y_0} \qquad i.e. \qquad f'(y_0).$$

- NOTE we are differentiating $\boldsymbol{f(y)}$ with respect to $\boldsymbol{y}$ here, so we get the rate of change of $\boldsymbol{f(y)}$ with respect to $\boldsymbol{y}$. In that case, $\boldsymbol{y}$ is being seen as an **independent** variable. We then *use that information* to determine what the behaviour of $\boldsymbol{y = y(t)}$ is for $\boldsymbol{y}$ values immediately above and below an equilibrium point, $\boldsymbol{y = y_0}$. *In this latter case, where the behaviour of* $\boldsymbol{y = y(t)}$ *is being deduced,* $\boldsymbol{y}$ *is a* **dependent** *variable and* $\boldsymbol{t}$ *the independent variable.* MAKE SURE YOU UNDERSTAND THIS!
- Understanding the points above is key to easily classifying the *equilibrium points* of an autonomous ODE as **stable**, **unstable**, or **semi-stable**, so make a special effort to understand the geometry and calculus behind the next few slides.

$$\boxed{f'(y_0) < 0}$$





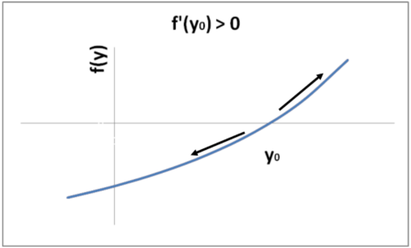
$f(y)$ is *decreasing* through 0 at $y = y_0$, therefore

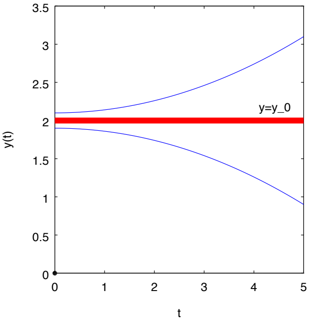
- $\hookrightarrow$ for solutions $y(t) < y_0$ we have $f(y) = \frac{dy}{dt} > 0$ so such solutions are *increasing* towards y_0 as $t \to \infty$;
- $\hookrightarrow$ Meanwhile for solutions $y(t) > y_0$ we have $f(y) = \frac{dy}{dt} < 0$ so such solutions are *decreasing* towards y_0 as $t \to \infty$.
- In summary, **solutions starting out near to y_0 move towards y_0,** $\cdots$

HENCE

$$\boxed{\begin{array}{c} f'(y_0) < 0 \\ \Downarrow \\ y_0 \text{ is a STABLE equilibrium point} \end{array}}$$

$$\boxed{f'(y_0) > 0}$$





$f(y)$ is *increasing* through 0 at $y = y_0$, therefore

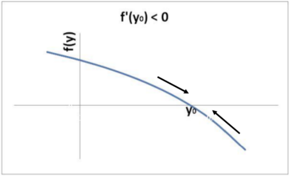
- $\hookrightarrow$ for solutions $y(t) < y_0$ we have $f(y) = \frac{dy}{dt} < 0$ so such solutions are *decreasing* away from y_0 as $t \to \infty$;
- $\hookrightarrow$ Meanwhile for solutions $y(t) > y_0$ we have $f(y) = \frac{dy}{dt} > 0$ so such solutions are *increasing* away from y_0 as $t \to \infty$.
- In summary, **solutions starting out near to y_0 move away from y_0,** $\cdots$

HENCE

$$\boxed{\begin{gathered} f'(y_0) > 0 \\ \Downarrow \\ y_0 \textbf{ is an UNSTABLE equilibrium point} \end{gathered}}$$

$$\boxed{f'(y_0) = 0}$$

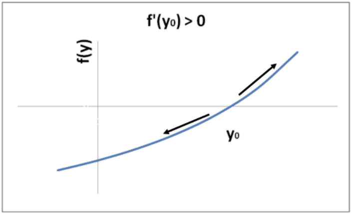
$\hookrightarrow$ This is an ambiguous case and requires a careful inspection of the ODE - in particular, of the **SIGN** **(+ve or −ve) of** $\boxed{f(y) = \frac{dy}{dt}}$ **immediately to the left and right of** $y = y_0$. There are **3** possibilities (recall always that $f(y_0) = 0$):



1. If $f(y)$ is positive to the left of y_0 and negative to the right of y_0, then $\boxed{y_0 \text{ is a } \textbf{STABLE} \text{ equilibrium point}}$.

- E.g. $y_0 = 0$ and $f(y) = -2y^5$.

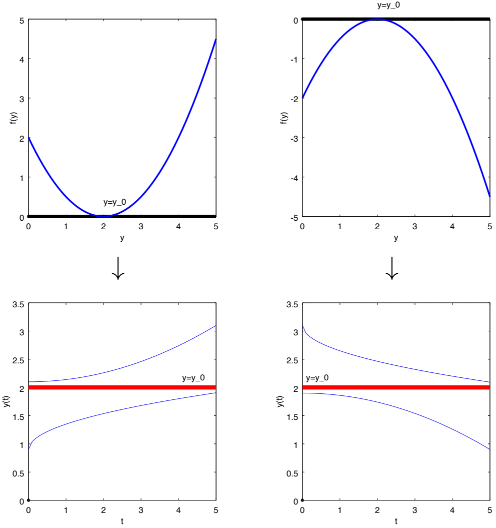
$$\boxed{f'(y_0) = 0}$$



2. If $f(y)$ is negative to the left of y_0 and positive to the right of y_0, then $\boxed{y_0 \text{ is an } \textbf{UNSTABLE} \text{ equilibrium point}}$.

- E.g. $y_0 = 0$ and $f(y) = y^3$.

$$\boxed{f'(y_0) = 0 \;\rightsquigarrow\; \text{case 3}}$$



3. **NEW CASE - SEMI-STABLE EQUILIBRIUM POINT** **If** $f'(y_0) = 0$ **and** $f(y)$ **is negative to the left AND right of** y_0 **OR** $f(y)$ **is positive to the left AND right of** y_0**, then** $\boxed{\boldsymbol{y_0} \textbf{ is an SEMI-STABLE equilibrium point}}$.

   In this case, starting values of y on *one side* of the equilibrium point, y_0, will approach y_0 as $t \to \infty$ **(stable)**, and starting values of y on the *other side* of y_0 will move away from y_0 as $t \to \infty$ **(unstable)**.

- E.g. $\boldsymbol{y_0 = 4}$ and $\boldsymbol{f(y) = (y-4)^2}$. Here, $\boldsymbol{f(y) = \frac{dy}{dt} > 0}$ on *both* sides of the equilibrium point, hence $\boldsymbol{y(t)}$ is **increasing** on both sides of the equilibrium point. So for $\boldsymbol{y < y_0}$, the solutions $\boldsymbol{y(t)}$ approach the equilibrium point **(stable)**, and for $\boldsymbol{y > y_0}$ the solutions $\boldsymbol{y(t)}$ move away from the equilibrium point **unstable**, hence $\boldsymbol{y_0}$ is an **SEMI-STABLE** equilibrium point.

- $\boxed{\text{EXAMPLE 8}}$ *(From a MATH1106 Tutorial)* Find all equilibrium points of the following differential equations, and use calculus to classify each equilibrium point as stable, unstable, or semi-stable:

(a) $\dfrac{dy}{dt} = (y^2 - 4)(y^2 - 25)(y + 2).$

(b) $\dfrac{dy}{dt} = e^{3y}.$

(c) $\dfrac{dy}{dt} = e^{3y} - e.$

(d) $\dfrac{dy}{dt} = (y - 4)^3 \ln(y^2 + 1).$

(a) Answer:

so that the equilibrium points are $y = -5, -2, 2, 5$.

$$f'(-5) = 630 > 0 \quad \Rightarrow \quad -5 \text{ is an UNSTABLE equilibrium point.}$$

$$f'(2) = -336 < 0 \quad \Rightarrow \quad 2 \text{ is a STABLE equilibrium point.}$$

$$f'(5) = 1470 > 0 \quad \Rightarrow \quad 5 \text{ is an UNSTABLE equilibrium point.}$$

hence $\boldsymbol{y = -2}$ *is a SEMISTABLE equilibrium point.*

(b) Answer:

(c) Answer:

(d) Answer:

Classifying Equilibrium Solutions via LINEARISATION *(Taylor series again!)*

↪ An alternative way to arrive at earlier conclusions concerning classifying an equilibrium solution, y_0, of $\dfrac{dy}{dt} = f(y)$ by looking at $f'(y_0)$, is to consider a **Taylor series** expansion about y_0 of a *nearby solution* $\boldsymbol{y(t) = y_0 + \eta(t)}$, where $\eta(t)$ represents a small perturbation of the equilibrium solution $y(t) = y_0$.

↪ First note $y(t) = y_0 + \eta(t)$ is also a solution of the differential equation therefore

$$\frac{d}{dt}(y_0 + \eta(t)) = \frac{d}{dt}(y_0) + \frac{d}{dt}(\eta(t)) = 0 + \frac{d\eta}{dt} = f(y_0 + \eta(t))$$

- So, taking the last equation in the sequence and using Taylor's theorem, provided $\eta(t)$ is smooth enough we have that

$$\frac{d\eta}{dt} = f(y_0+\eta(t)) = f(y_0)+f'(y_0)\eta(t)+f''(y_0)\frac{\eta(t)^2}{2!}+\textit{Higher Order Terms}$$

$\hookrightarrow$ Given that $f(y_0) = 0$ and if $\eta(t)$ is small enough for us to ignore all nonlinear higher order terms and truncate the Taylor series after the linear term *(hence the word "linearisation")*, then we see that (approximately) $\eta(t)$ satisfies the linear and separable ODE

$$\frac{d\eta}{dt} = f'(y_0)\eta(t)$$

which can be easily solved to get $\eta(t) = Ce^{f'(y_0)t}$.

- From this solution, it is clear that for the nearby solution $y(t) = y_0 + \eta(t) = y_0 + e^{f'(y_0)t}$ to equilibrium solution $y(t) = y_0$, if $f'(y_0) < 0$ then $\lim_{t\to\infty} y(t) = y_0$ hence the equilibrium solution $y(t) = y_0$ is STABLE, whereas if $f'(y_0) > 0$ then $\lim_{t\to\infty} y(t) = \infty$ hence the equilibrium solution $y(t) = y_0$ is UNSTABLE.
- **NOTE** we also see that if $f'(y_0) = 0$ then the nonlinear terms in the Taylor series expansion determine the stability properties of equilibrium solution $y(t) = y_0$.

Direction Fields - a qualitative look at solutions

- It is important that you understand **(1)** what a direction field for a given DE is, **(2)** how to create one, and **(3)** what it tells you about solutions to that equation.

- Consider the general first order ODE $\frac{dx}{dt} = f(t, x)$. What is the dependent variable? .

- In the $t - x$ plane, for each *specific* ordered pair (t_0, x_0), $\frac{dx}{dt}|_{(t_0,x_0)} = f(t_0, x_0)$ is the **slope/gradient/derivative** of a solution curve at the point (t_0, x_0). So a short line segment at (t_0, x_0) with slope $\frac{dx}{dt}|_{(t_0,x_0)} = f(t_0, x_0)$ will *roughly* look like a small portion of the solution curve at the point (t_0, x_0).

- **Definition:** If we compute $f(t_0, x_0)$ and then draw a short line segment at (t_0, x_0) of slope $f(t_0, x_0)$ **for a large number of points (t_0, x_0)** in the $t - x$ plane, we get a **DIRECTION FIELD** for the DE $\frac{dx}{dt} = f(t, x)$.

- A direction field for a DE gives a good idea of how the graphs of solutions to the DE look. *In other words, it gives a good qualitative idea of the behaviour of solutions to the DE*.

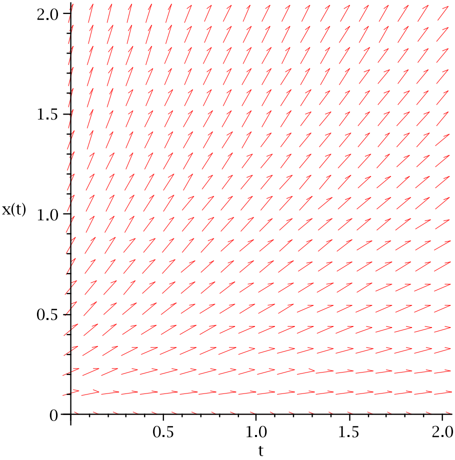
↪ **Online Direction Field Drawer**
http://www.math.psu.edu/cao/DFD/Dir.html

↪ EXAMPLE 9 $\dfrac{dx}{dt} = \dfrac{2x}{1+t}$. Draw a direction field for $0 \le t \le 2$ and $0 \le x \le 2$.

↪ A few sample calculations: at $t = 0, x = 0$ we get $\frac{dx}{dt} = 0$. At $t = 2, x = 1$ we have $\frac{dx}{dt} = \frac{2}{3}$. At $t = 1, x = 2$ we have $\frac{4}{2} = 2$. See table below for more:

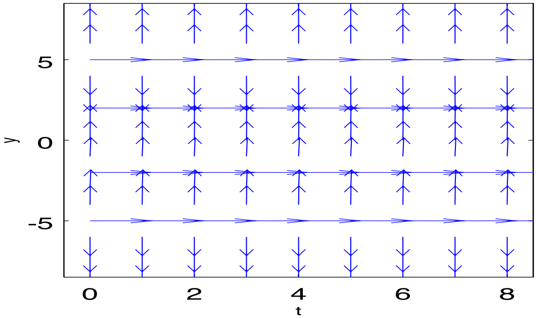
$\frac{dx}{dt} = \frac{2x}{1+t}$, $t = 0 \ldots 2, x = 0 \ldots 2$

$t \rightarrow$ $x \downarrow$	**0**	**0.5**	**1**	**1.5**	**2**
0	0	0	0	0	0
0.5	1	2/3	1/2	2/5	1/3
1	2	4/3	1	4/5	2/3
1.5	3	2	3/2	6/5	1
2	4	8/3	2	8/5	4/3



$t \rightarrow$ $x \downarrow$	**0**	**0.5**	**1**	**1.5**	**2**
0	0	0	0	0	0
0.5	1	2/3	1/2	2/5	1/3
1	2	4/3	1	4/5	2/3
1.5	3	2	3/2	6/5	1
2	4	8/3	2	8/5	4/3

- Direction fields can give us insight into **equilibrium solutions** of DEs.
- E.g., From EXAMPLE 8(a), the direction field below clearly shows the four equilibrium solutions and makes it easy to classify them:



$y = -5 \rightsquigarrow$ unstable; $y = -2 \rightsquigarrow$ semistable;

$y = 2 \rightsquigarrow$ stable; $y = 5 \rightsquigarrow$ unstable

↪ Optionally and preferably, you can also use MATLAB to draw direction fields, using the `quiver()` function.

- For more on this see **Tutorial 2**.

Numerical Solutions of First Order IVPs

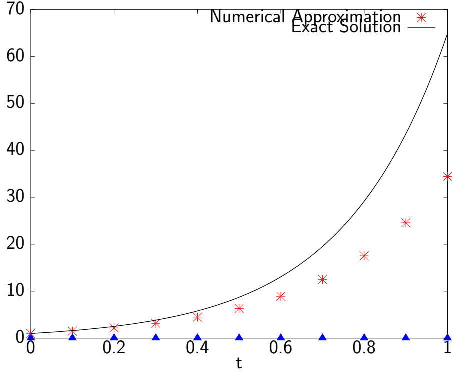
- In the interest of time, we will cover only 3 methods:
    1. Euler's method - *for simplicity*
    2. Heun's (Improved Euler) method - *relatively simply and fairly accurate, error* $O(h^2)$
    3. Runge-Kutta (4th order) - *very accurate, error* $O(h^4)$.
- For practical purposes, only the last two will be used much since they are far more accurate that Euler's method.
- We will later generalise them to deal with systems of ODEs.
- These numerical methods will be helpful as tools to investigate numerical solutions to IVPs and systems of IVPs which are difficult or impossible to solve analytically.

INTRODUCTION

- We seek *approximate solutions* to the general first order IVP

$$\textit{Find } y(t) \textit{ such that } \frac{dy}{dt} = f(t, y), \quad t \in [t_0, T], \quad y(t_0) = y_0.$$

- The numerical methods we will explore here to approximate the solutions to ODEs work by replacing the *continuous* problem of finding a continuous function $y(t)$ by the *discrete* problem of approximating $y(t)$ at $N + 1$ (usually) equally-spaced points. If we then wish to reconstruct an approximation to the *entire* function $y(t)$, we typically just join the points (t_i, y_i) by line segments to obtain a *piece-wise linear* approximation to $y(t)$ on all of $[t_0, T]$ $\rightsquigarrow$ see the next page for an illustration of this
- **As always, NOTATION and LANGUAGE will be very important in what follows; pay close attention to it and ensure that you learn and understand the notation and language used!**



Here the unknown function $y(t)$ is approximated at only 11 points, $\boxed{t_0 = 0,\ t_1 = 0.1,\ t_2 = 0.2,\ \ldots,\ t_{10} = 1}$ *(indicated by the* $\blacktriangle$ *symbols on the* $t-$ *axis)*, and the approximate value of the function at each t_i is indicated by the $*$ symbols.

NOTATION

↪ In general, when approximating the solution to

$$\frac{dy}{dt} = f(t, y), \quad t \in [t_0, T], \quad y(t_0) = y_0.$$

we adopt the following notation conventions:

⇝ The interval $[t_0, T]$ is broken into N subintervals whose $N+1$ points are denoted $t_0, t_1, t_2, \ldots, t_N$, and with

$$h = \frac{T - t_0}{N}$$

the t_is are the evenly-spaced points *(the only case we will consider in depth in this course)* $t_i = t_0 + ih, \quad i = 0, 1, 2, \ldots, N.$

⇝ Since a numerical method gives only an *approximation* to the true solution function $y(t)$ at certain t values, we use a different notation to refer to this approximate solution:

$$Y_i \text{ or } y_i \quad \approx \quad y(t_i) \quad \text{for } i = 0, 1, 2, \ldots, N.$$

NOTATION ALERT!!!

↪ One of the *(annoying)* features of the chosen programming language for this course, MATLAB, is that all vectors (more generally, all arrays) *MUST* have their indexing start at $\boxed{\mathbf{1}}$ and not $\boxed{\mathbf{0}}$ *(indexing starting at* 0 *is allowed and is the default in most other relevant programming languages)*.

- So if we have a MATLAB vector t, its first entry must be $\boldsymbol{t(1)}$, not $\boldsymbol{t(0)}$.

⇝ One negative consequence of this is that we cannot translate the notation just introduced *directly* into MATLAB by mapping $t_i \mapsto$ `t(i)` and $y_i \mapsto$ `y(i)` for $i = 0, 1, 2, \ldots, N$.

≻ Instead, we must have $\boxed{t_i \mapsto \texttt{t(i+1)}}$ and $\boxed{y_i \mapsto \texttt{y(i+1)}}$ for $i = 0, 1, 2, \ldots, N$.

NOTATION ALERT (continued) !!!

- One simple solution is to use the **1 to N+1 indexing** throughout the lecture notes for all formulae, so that they can easily be translated into MATLAB. The disadvantage of this approach is that the **0 to N indexing** is what is used in most textbooks, and is by far more common and intuitive than the **1 to N+1 indexing**.

- **THEREFORE, I have chosen to use the traditional $\boxed{0 \text{ to } N}$ indexing in these lecture notes**.

  - As a consequence, you will have to do the index mapping $\boxed{i \mapsto i+1}$ whenever translating formulae for different numerical methods from the notes to MATLAB programs.
  - *For example, a loop that runs from* 0 *to* N *in the notes would have to run from* 1 *to* $N+1$ *in MATLAB*.

- However, to make life easier, I will endeavour to always give the main formula for a numerical method in the standard form with the **0 to N indexing**, *THEN* in MATLAB form with **1 to N+1 indexing** (and vector notation).

Euler's Method

↪ We begin with the simplest rule for approximating the solution to the general first order IVP $\frac{dy}{dt} = f(t, y), \quad t \in [t_0, T], \quad y(t_0) = y_0$.:

Euler's Method

$$\boldsymbol{y_0 = y(t_0)} \text{ THEN}$$

$$y_{i+1} = y_i + hf(t_i, y_i) \text{ for } i = 0, 1, 2 \ldots, N-1.$$

In MATLAB syntax, this would be

Euler's Method (MATLAB version)

$$\boldsymbol{t(1) = t_0} \text{ and } \boldsymbol{t = t_0 : h : T},$$
$$\boldsymbol{y(1) = y(t(1)) = y_0} \text{ THEN}$$

$$y(i+1) = y(i) + hf(t(i), y(i)) \text{ for } i = 1, 2, 3 \ldots, N.$$

- See Tutorial 2 for programs corresponding to Euler's method, Heun's Method, and the fourth order Runge-Kutta method.

    - The method is derived from truncating a Taylor series expansion of the (unknown) solution function $y(t)$.

REMINDER $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ then in Euler's method $y_0 = y(t_0)$ and $y_{i+1} = y_i + hf(t_i, y_i)$ for $i = 0, 1, 2 \ldots, N-1$

- **DERIVATION** Euler's method is easy to derive from a Taylor series expansion of the solution in which the quadratic and higher order terms are ignored. Assuming that the solution function $y(t)$ of the standard first order IVP is C^2 on $[t_0, T]$ *(i.e., $y(t)$, $y'(t)$ and $y''(t)$ are continuous on $[t_0, T]$)*, then

$$\begin{aligned} y(t+h) &= y(t) + hy'(t) + \frac{1}{2}y''(t)h^2 + O(h^3) \\ &= y(t) + hf(t, y) + \frac{1}{2}y''(t)h^2 + O(h^3) \end{aligned}$$

$\hookrightarrow$ In particular, Euler's method follows from the above derivation easily by letting $t = t_i = t_0 + ih$ so that

$$y(t_i + h) = \underbrace{y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i))}_{\text{"Euler's Method"}} + \underbrace{\frac{1}{2}y''(t_i^*)h^2}_{\text{Local Truncation Error (LTE)}},$$

where $t_i^* \in [t_i, t_i + h]$ $\forall i = 0, 1, 2, \ldots N-1$ by **Taylor's theorem**.

$\hookrightarrow$ For $i = 0$, the above method, $y_1 \approx y_0 + hf(t_0, y_0)$ *is* Euler's method. For subsequent timesteps, since in general $y_{i+1} \neq y(t_{i+1})$, Euler's method is obtained by replacing $\boldsymbol{y(t_{i+1})}$ and $\boldsymbol{y(t_i)}$ in the above equation by their Euler's method approximations, $\boldsymbol{y_{i+1}}$ and $\boldsymbol{y_i}$. NOTE LTE $O(h^2) \Rightarrow$ Global Error $O(h)$.

Heun's Method *(or The Improved Euler Method)*

$\hookrightarrow$ **DERIVATION** The idea here is to start with Euler's method

$$y_{i+1} = y_i + hf(t_i, y_i)$$

and *TRY* to replace the $f(t_i, y_i)$ by the AVERAGE of f evaluated at (t_i, y_i) and (t_{i+1}, y_{i+1}).

- What is the flaw in this plan???

- What is a way around this problem???

:

$$f(t_{i+1}, y_{i+1}) \text{ is approximated by } f(t_{i+1}, y_i + hf(t_i, y_i)).$$

↪ **IMPLEMENTATION** Heun's method can be described in a one-step or two-step manner:

Heun's Method

ONE-STEP	TWO-STEP
$y_0 = y(t_0)$ THEN	$y_0 = y(t_0)$ THEN
	$\tilde{y}_{i+1} = y_i + hf(t_i, y_i)$ AND
$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))\right]$	$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})\right]$
for $i = 0, 1, 2 \ldots, N-1$.	for $i = 0, 1, 2 \ldots, N-1$

↪ In MATLAB syntax this would be: **Heun's Method (MATLAB version)**

ONE-STEP	TWO-STEP
$t = t_0 : h : T$, AND $y(1) = y(t(1)) = y_0$ THEN	
$y(i+1) =$ $y(i) + \frac{h}{2}\left[f(t(i), y(i)) + f(t(i+1), y(i) + hf(t(i), y(i)))\right]$	$ytemp = y(i) + hf(t(i), y(i))$ AND $y(i+1) = y(i) + \frac{h}{2}\left[f(t(i), y(i)) + f(t(i+1), ytemp)\right]$
for $i = 1, 2, \ldots, N$	

The Fourth-Order Runge-Kutta Method (RK4)

$$\begin{aligned}
k_1 &= hf(t_n, y_n)\\
k_2 &= hf(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)\\
k_3 &= hf(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)\\
k_4 &= hf(t_n + h, y_n + k_3)\\
\boldsymbol{y_{n+1}} &= \boldsymbol{y_n + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4}
\end{aligned}$$

EXAMPLE 10: Solving $\frac{dy}{dt} = -\frac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 5$ subintervals (so $h = 0.2$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

i	TIME	Y_i (EULER)	$y(t_i)$ (EXACT)	ERROR
0	1.000000	12.000000	12.000000	0.0000000000
1	1.200000	10.800000	10.954451	0.1544511501
2	1.400000	9.900000	10.141851	0.2418510567
3	1.600000	9.192857	9.486833	0.2939758376
4	1.800000	8.618304	8.944272	0.3259683386
5	2.000000	8.139509	8.485281	0.3457724457

i	TIME	Y_i (HEUN)	$y(t_i)$ (EXACT)	ERROR
0	1.000000	12.000000	12.000000	0.0000000000
1	1.200000	10.950000	10.954451	0.0044511501
2	1.400000	10.135268	10.141851	0.0065831996
3	1.600000	9.479190	9.486833	0.0076427302
4	1.800000	8.936112	8.944272	0.0081602678
5	2.000000	8.476895	8.485281	0.0083865803

i	TIME	Y_i (RK4)	$y(t_i)$ (EXACT)	ERROR
0	1.000000	12.000000	12.000000	0.0000000000
1	1.200000	10.954442	10.954451	0.0000090013
2	1.400000	10.141839	10.141851	0.0000119113
3	1.600000	9.486820	9.486833	0.0000127666
4	1.800000	8.944259	8.944272	0.0000128513
5	2.000000	8.485269	8.485281	0.0000126333

Compare the errors at the final time with the errors in the following slides where the same problem is done with $N = 10 \Rightarrow h = 0.1$.

Reminder: solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1, 2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

i	**TIME**	Y_i **(EULER)**	$y(t_i)$ **(EXACT)**	**ERROR**
0	1.000000	12.000000	12.000000	0.0000000000
1	1.100000	11.400000	11.441551	0.0415510709
2	1.200000	10.881818	10.954451	0.0726329683
3	1.300000	10.428409	10.524696	0.0962871408
4	1.400000	10.027316	10.141851	0.1145346232
5	1.500000	9.669198	9.797959	0.1287609816
6	1.600000	9.346891	9.486833	0.1399415906
7	1.700000	9.054801	9.203580	0.1487788322
8	1.800000	8.788483	8.944272	0.1557885535
9	1.900000	8.544359	8.705715	0.1613561825
10	2.000000	8.319507	8.485281	0.1657741033

Reminder: solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

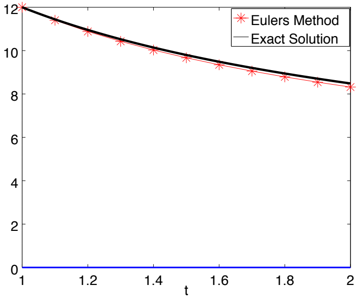
i	**TIME**	Y_i **(HEUN)**	$y(t_i)$ **(EXACT)**	**ERROR**
0	1.000000	12.000000	12.000000	0.0000000000
1	1.100000	11.440909	11.441551	0.0006419800
2	1.200000	10.953370	10.954451	0.0010807989
3	1.300000	10.523310	10.524696	0.0013860946
4	1.400000	10.140250	10.141851	0.0016009727
5	1.500000	9.796206	9.797959	0.0017530864
6	1.600000	9.484972	9.486833	0.0018607202
7	1.700000	9.201644	9.203580	0.0019362809
8	1.800000	8.942284	8.944272	0.0019883768
9	1.900000	8.703692	8.705715	0.0020230975
10	2.000000	8.483237	8.485281	0.0020448245

Reminder: solving $\frac{dy}{dt} = -\frac{1}{2t}y$, $t \in [1, 2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

i	**TIME**	Y_i **(RK4)**	$y(t_i)$ **(EXACT)**	**ERROR**
0	1.000000	12.000000	12.000000	0.0000000000
1	1.100000	11.441551	11.441551	0.0000003598
2	1.200000	10.954451	10.954451	0.0000005624
3	1.300000	10.524696	10.524696	0.0000006781
4	1.400000	10.141850	10.141851	0.0000007436
5	1.500000	9.797958	9.797959	0.0000007792
6	1.600000	9.486832	9.486833	0.0000007967
7	1.700000	9.203579	9.203580	0.0000008028
8	1.800000	8.944271	8.944272	0.0000008018
9	1.900000	8.705714	8.705715	0.0000007963
10	2.000000	8.485281	8.485281	0.0000007881

Reminder: solving $\dfrac{dy}{dt} = -\dfrac{1}{2t}y$, $t \in [1,2]$, $y(1) = 12$ using $N = 10$ subintervals (so $h = 0.1$) $\rightsquigarrow$ EXACT SOLUTION, $y(t) = 12t^{-1/2}$.

- Only an Euler's method plot is shown below since the other methods are sufficiently accurate for their graphs to be indistinguishable from the exact solution graph.



- Note how with Euler's method *(Error $O(h)$)* halving of the step size causes the error in the final step to reduce by about one half, with Heun's method *(Error $O(h^2)$)* the reduction in error is by about one quarter, and with the 4th order Runge-Kutta method *(Error $O(h^4)$)* the reduction in error is by about one sixteenth.

  Students familiar with big O notation should know why.

HOMEWORK:

- Review the material in this lecture and do Tutorial 2. Come to class with questions about any aspect of this review which you find particularly difficult.
- See Tutorial 1 for a quick introduction to Matlab. Since it will be used as a tool in this course, it is useful to have some familiarity with it.
- Have good calculus and linear algebra books nearby.

APPENDIX - Vocabulary/Notation

$\hookrightarrow$ Becoming familiar with the relevant vocabulary and notation is crucial to developing proficiency in mathematics and its applications. Test yourself to see if you have a good understanding of the following, all mentioned/defined earlier in this lecture (and in no particular order):

- Initial value problem.
- Ordinary differential equation.
- Equilibrium solution.
- Separable ODE.
- Stable, unstable, semistable equilibrium solution.
- Partial differential equation.
- First order ODE.
- Direction field.
- Critical point.
- Autonomous first order ODE.
- Separation of variables.
- Linear first order ODE.
- Equilibrium point.
- Nonlinear ODE.