MATH1134 - Eigenvalue/Eigenvector Supplementary Lecture

Definitions and Conventions

DEFINITIONS: If $A_{n\times n}$ is a square matrix then a *non-zero* vector $\vec{x}$ is called an **eigenvector** of A if $A\vec{x}$ is a scalar multiple of $\vec{x}$. In other words

$$\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$$

for some scalar λ. That scalar λ is called an **eigenvalue** of A and $\vec{x}$ is said to be an eigenvector of A corresponding to λ *(always think of an eigenvalue-eigenvector PAIR)*.

- $\hookrightarrow$ So multiplying an eigenvector $\vec{x}$ of A on the left by A simply stretches or compresses $\vec{x}$ by a factor of $|\lambda|$ and reverses the direction of $\vec{x}$ if $\lambda < 0$.
- $\hookrightarrow$ NOTATION: It is conventional to use λ to represent eigenvalues.

$\hookrightarrow$ $\boxed{\text{EXAMPLE 1}}$ **(a)** If $A = \begin{bmatrix} 1 & 6 \\ -2 & -6 \end{bmatrix}$ has eigenvector $\vec{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, find the corresponding eigenvalue. **(b)** If $\lambda = -2$ is another eigenvalue of A, find a corresponding eigenvector.

(a) We seek λ such that

$$\begin{bmatrix} 1 & 6 \\ -2 & -6 \end{bmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} -3 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} -3+12 \\ 6-12 \end{pmatrix} = \begin{pmatrix} -3\lambda \\ 2\lambda \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 9 \\ -6 \end{pmatrix} = \begin{pmatrix} -3\lambda \\ 2\lambda \end{pmatrix}$. So $9 = -3\lambda$ or $-6 = 2\lambda$ which both imply that $\boxed{\lambda = -3}$ (CHECK!).

(b) We seek a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $A\vec{x} = -2\vec{x}$:

$$\begin{bmatrix} 1 & 6 \\ -2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x_1 + 6x_2 \\ -2x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}.$$

So $x_1 + 6x_2 = -2x_1 \Rightarrow \boxed{3x_1 + 6x_2 = 0}$.

And $-2x_1 - 6x_2 = -2x_2 \Rightarrow \boxed{-2x_1 - 4x_2 = 0}$.

Combining the last two boxed equations, we see

$3x_1 + 6x_2 = -2x_1 - 4x_2 \Rightarrow 5x_1 + 10x_2 = 0$ or $\boxed{x_1 = -2x_2}$, and any eigenvector for eigenvalue -2 is of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix}$. E.g. $x_2 = 1$ yields $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

`(CHECK IT IS AN EIGENVECTOR WITH EIGENVALUE -2!).`

$\hookrightarrow$ So from the last example, we see that an eigenvalue can have infinitely many eigenvectors. In fact, this is always the case. Each eigenvalue λ of a matrix A has infinitely many corresponding eigenvectors $\vec{x}$ (so $A\vec{x} = \lambda\vec{x}$). *ANY* scalar multiple of an eigenvector of A corresponding to eigenvalue λ is another eigenvector of A.

PROOF: If α is a non-zero scalar and λ is an eigenvalue of A with corresponding eigenvector $\vec{x}$, then $A\vec{x} = \lambda\vec{x}$. Therefore, it follows that $A\alpha\vec{x} = \alpha(A\vec{x}) = \alpha(\lambda\vec{x}) = \lambda\alpha\vec{x}$. In summary, $A(\alpha\vec{x}) = \lambda(\alpha\vec{x})$ so that the vector $\alpha\vec{x}$ is an eigenvector of A with eigenvalue λ.

How to Find Eigenvalues and Eigenvectors of a Given Matrix, A

$\hookrightarrow$ Always find the eigenvalues first, then the eigenvectors corresponding to those eigenvalues.

$\hookrightarrow$ Recall the eigenvalue-eigenvector equation is $A\vec{x} = \lambda\vec{x}$. Assume A is a known matrix and λ and $\vec{x}$ are to be found.

$\rightsquigarrow$ It is convenient to re-arrange the equation as follows:

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} = \lambda\mathrm{I}\vec{x} \iff A\vec{x} - \lambda\mathrm{I}\vec{x} = \vec{0}$$

$$\iff (A - \lambda\mathrm{I})\vec{x} = \vec{0}. \tag{1}$$

$\rightsquigarrow$ Recall from earlier (Lecture 2) that if $(A - \lambda I)$ is non-singular, then the system (1) has only *one* solution vector - clearly, $\vec{x} = \vec{0}$. But recalling that $\vec{0}$ cannot, by definition, be an *eigenvector*, we require that system (1) has *infinitely many solutions*. This occurs when $\det(A - \lambda\mathrm{I}) = 0$ *(or equivalently, when $A - \lambda\mathrm{I}$ is a singular matrix).*

$\mapsto$ Since A and I are given, the only unknown in the equation $\det(A - \lambda\mathrm{I}) = 0$ is the eigenvalue(s), λ. **Hence we solve $\det(A - \lambda\mathrm{I}) = 0$ to find the eigenvalue(s),** λ.

- $\boxed{\text{DEFINITIONS:}}$ The equation $\det(A - \lambda \mathrm{I}) = 0$ is called the **characteristic equation** of A, and the expression $\det(A - \lambda \mathrm{I})$ is a degree n polynomial (if A is $n \times n$) called the **characteristic polynomial** of A.
  - So the eigenvalues of A are simply *the roots of its characteristic polynomial*, or equivalently, *the solutions of its characteristic equation*.
  - NOTE then that if matrix A is $n \times n$, it will have at most n distinct roots (some of which might be complex numbers).
- Once the eigenvalues of A are found, simply substitute each into $(A - \lambda \mathrm{I})\vec{x} = \vec{0}$ (or if you prefer, into the equivalent equation $A\vec{x} = \lambda\vec{x}$) and solve for eigenvector(s) $\vec{x}$ - just as in EXAMPLE 1(b).

$\hookrightarrow$ $\boxed{\text{EXAMPLE 2:}}$ Find all eigenvalues and, for each eigenvalue, a corresponding eigenvector for $A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$.

$\mapsto$ $\boxed{\text{ANSWER:}}$ First the eigenvalues: Solve $\det(A - \lambda \mathrm{I}) = 0$. So

$$\begin{vmatrix} 2-\lambda & 5 \\ 6 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\lambda)(1-\lambda) - 5(6) = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 28 = 0 \quad \Rightarrow \quad (\lambda - 7)(\lambda + 4) = 0.$$

So the two eigenvalues are $\boxed{\lambda_1 = 7}$ and $\boxed{\lambda_2 = -4}$.

$\mapsto$ To find eigenvector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponding to eigenvalue $\lambda_1 = 7$, we solve

$$(A - 7\mathrm{I})\vec{x} = \vec{0} \text{ or } \begin{bmatrix} 2-7 & 5 \\ 6 & 1-7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -5 & 5 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $-5x_1 + 5x_2 = 0$, $\quad 6x_1 - 6x_2 = 0$. Both equations imply that $x_1 = x_2$ so a typical eigenvector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$. So, for example, setting $x_2 = 1$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue $\lambda_1 = 7$. *(CHECK!)*

REMINDER: $A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$ has eigenvalues 7, -4.

- An eigenvector $\vec{x} = [x_1, x_2]^t$ corresponding to eigenvalue $\lambda_2 = -4$ satisfies $(A + 4\mathrm{I})\vec{x} = \vec{0}$ or

$$\begin{bmatrix} 2+4 & 5 \\ 6 & 1+4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 6 & 5 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So we have two equations which are the same: $6x_1 + 5x_2 = 0 \Rightarrow x_1 = -\frac{5}{6}x_2$. So a typical eigenvector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{6}x_2 \\ x_2 \end{bmatrix}$. So, for example, taking $x_2 = 6$ we get the eigenvector

$$\vec{x} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$\hookrightarrow$ EXAMPLE 3: Find all eigenvalues and, for each eigenvalue, a corresponding eigenvector for $A = \begin{bmatrix} 2 & 0 & 0 \\ -4 & -5 & 0 \\ 1 & 0 & 4 \end{bmatrix}$.

$\mapsto$ ANSWER: Because $A - \lambda I$ is a (lower) triangular matrix, the determinant is easy to compute: being just the product of the entries on the main diagonal *(it will NOT always be so easy; see EXAMPLE 4 next)*. So $0 = \det(A - \lambda \mathrm{I}) =$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -4 & -5-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} \quad \Rightarrow \quad (2-\lambda)(-5-\lambda)(4-\lambda) = 0.$$

So clearly the three eigenvalues are

$$\boldsymbol{\lambda_1 = 2,}$$

$$\boldsymbol{\lambda_2 = -5,} \quad \textbf{and}$$

$$\boldsymbol{\lambda_3 = 4.}$$

$\hookrightarrow$ Next, we find an eigenvector corresponding to eigenvalue $\lambda_1 = 2$: We seek $\vec{x} = [x_1, x_2, x_3]^T$ such that $(A - 2\mathrm{I})\vec{x} = \vec{0}$, or equivalently

$$\begin{bmatrix} 0 & 0 & 0 \\ -4 & -7 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system is already in (lower) triangular form, so there is no real need for Gaussian elimination. So, other than the first row which just tells us $0 = 0$, we have two equations: **(1)** $x_1 + 2x_3 = 0 \Rightarrow \boxed{x_1 = -2x_3}$ and **(2)** $-4x_1 - 7x_2 = 0 \Rightarrow \boxed{x_2 = -\frac{4}{7}x_1}$. But we use the previous equation, $x_1 = -2x_3$, to further simplify the last result to $\boxed{x_2 = \frac{8}{7}x_3}$. So a typical eigenvector is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ \frac{8}{7}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ \frac{8}{7} \\ 1 \end{bmatrix}.$$

So, for example, when $x_3 = 7$, an eigenvector of A corresponding to eigenvalue $\lambda_1 = 2$ is

$$\begin{bmatrix} \mathbf{-14} \\ \mathbf{8} \\ \mathbf{7} \end{bmatrix}.$$ **CHECK!**

$\hookrightarrow$ Next, we find an eigenvector corresponding to eigenvalue $\lambda_2 = -5$:
We seek $\vec{x} = [x_1, x_2, x_3]^T$ such that $(A + 5\mathrm{I})\vec{x} = \vec{0}$, or equivalently

$$\begin{bmatrix} 7 & 0 & 0 \\ -4 & 0 & 0 \\ 1 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system is already in (lower) triangular form, so there is no real need for Gaussian elimination. The first two equations state that $\boxed{x_1 = 0}$ and the last equation $x_1 + 9x_3 = 0 \Rightarrow \boxed{x_3 = 0}$. So a typical eigenvector is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

So, for example, when $x_2 = 1$, an eigenvector of A corresponding to eigenvalue $\lambda_2 = -5$ is

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}. \quad \textbf{CHECK!}$$

$\hookrightarrow$ Finally, we find an eigenvector corresponding to eigenvalue $\lambda_3 = 4$: We seek $\vec{x} = [x_1, x_2, x_3]^T$ such that $(A - 4\mathrm{I})\vec{x} = \vec{0}$, or equivalently

$$\begin{bmatrix} -2 & 0 & 0 \\ -4 & -9 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system is already in (lower) triangular form, so there is no real need for Gaussian elimination. The first and last equations state that $\boxed{x_1 = 0}$ and the second equation $-4x_1 - 9x_2 = 0 \Rightarrow \boxed{x_2 = 0}$. So a typical eigenvector is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, for example, when $x_3 = 1$, an eigenvector of A corresponding to eigenvalue $\lambda_3 = 4$ is

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}. \quad \textbf{CHECK!}$$

$\hookrightarrow$ EXAMPLE 4: Find all eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

$\mapsto$ ANSWER: Using cofactor expansion along the first row,
$0 = \det(A - \lambda I) =$

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = -\lambda[(2-\lambda)(3-\lambda) - 1(0)] - 0[3-\lambda-1] - 2[1(0) - (2-\lambda)]$$

$$= -\lambda(\lambda^2 - 5\lambda + 6) - 0 - 2(\lambda - 2) \quad = \quad -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0 \Rightarrow$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \quad \text{or} \quad (\lambda - 1)(\lambda - 2)^2 = 0.$$

So the *two* distinct eigenvalues of A are $\boxed{\lambda_1 = 1}$ and $\boxed{\lambda_2 = 2}$.

$\hookrightarrow$ NOTE A is 3×3 and we have *fewer* than 3 distinct eigenvalues (however, if we count an eigenvalue as often as its *multiplicity*, we still get 3). There is the *danger* then that we will have only *two* families of linearly independent eigenvectors **[see 3 pages after this for discussion of linear independence]** ... in fact, you will see next that we still get 3 distinct families of linearly independent eigenvectors in this case *(but that does not always happen!)*.

$\hookrightarrow$ Next, we find an eigenvector corresponding to eigenvalue $\lambda_1 = 1$: We seek $\vec{x} = [x_1, x_2, x_3]^T$ such that $(A - 1\mathrm{I})\vec{x} = \vec{0}$, or equivalently

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We use Gaussian elimination: $R3 \mapsto R3 + R1$ and $R2 \mapsto R2 + R1$ lead to the equivalent (upper-triangular) system

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, other than the last row which just tells us $0 = 0$, we have two equations: **(1)** $x_2 - x_3 = 0 \Rightarrow \boxed{x_2 = x_3}$ and **(2)** $-x_1 - 2x_3 = 0 \Rightarrow \boxed{x_1 = -2x_3}$.

So a typical eigenvector is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. So, for example, when $x_3 = 1$, an eigenvector of A corresponding to eigenvalue $\lambda_1 = 1$ is

$$\begin{bmatrix} \mathbf{-2} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} \quad \textbf{(CHECK!)}.$$

$\hookrightarrow$ Next, we seek eigenvectors corresponding to eigenvalue $\lambda_2 = 2$ *(of multiplicity 2)*: We seek $\vec{x} = [x_1, x_2, x_3]^T$ such that $(A - 2\mathrm{I})\vec{x} = \vec{0}$, or equivalently

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We use Gaussian elimination: $R3 \mapsto R3 + \frac{1}{2}R1$ and $R2 \mapsto R2 + \frac{1}{2}R1$ lead to the equivalent (upper-triangular) system

$$\begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The last two rows tell us $0 = 0$, and the first row says: $-2x_1 - 2x_3 = 0 \Rightarrow \boxed{x_1 = -x_3}$ *(and x_2 is independent of x_1 and x_3!)*. So a typical eigenvector is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. For example, when $x_3 = 1$ and $x_2 = 0$, an eigenvector of A corresponding to eigenvalue $\lambda_2 = 2$ is $\begin{bmatrix} \mathbf{-1} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ **(CHECK!).** And when $x_3 = 0$ and $x_1 = 1$ an eigenvector of A corresponding to eigenvalue $\lambda_2 = 2$ is $\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$ **(CHECK!).**

QUICK REVIEW OF LINEARLY INDEPENDENT VECTORS

- Recall a **linear combination** of the (known) vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is simply any (finite) sum $k_1\vec{v_1} + k_2\vec{v_2} + \ldots + k_n\vec{v_n}$ where $k_1, k_2, \ldots, k_n$ are scalars.
- For known vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, the vector equation

$$k_1\vec{v_1} + k_2\vec{v_2} + \ldots + k_n\vec{v_n} = \vec{0} \tag{2}$$

always has the solution $k_1 = 0, \quad k_2 = 0, \; \ldots \; k_n = 0$. If this is the only solution, the vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are said to be **linearly independent**. If there are other solutions (where at least one $k_i \neq 0,\ i = 1, \ldots, n$) then the vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are said to be **linearly dependent**.

- $\boxed{\text{EQUIVALENT DEFINITIONS:}}$ **(a)** $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are **linearly independent** if *none* of the vectors can be written as a linear combination of the others.

(b) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are **linearly dependent** if *at least one* of the vectors can be written as a linear combination of the others.

- $\hookrightarrow$ A set containing only 2 vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if and only if one vector is a scalar multiple of the other. And if not, the set is linearly independent.
- $\hookrightarrow$ For example, in $\mathbb{R}^4$ the vectors $\vec{v}_1 = (-4, 1, 0, 3)$, $\vec{v}_2 = (1, -2, 3, -4)$, and $\vec{v}_3 = (-5, -4, 9, -6)$ are linearly dependent since $2\vec{v}_1 + 3\vec{v}_2 - \vec{v}_3 = \vec{0}$ (CHECK!)
- $\hookrightarrow$ In $\mathbb{R}^3$, $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (0, 1, 0)$, and $\vec{v}_3 = (0, 0, 1)$ are obviously(???) *linearly independent*. But if we add the vector $\vec{v}_4 = (-3, 4, 7)$ to the set, it becomes *linearly dependent* since $\vec{v}_4 = -3\vec{v}_1 + 4\vec{v}_2 + 7\vec{v}_3$.
- $\hookrightarrow$ One way to check if a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent (or linearly dependent) is to try to solve the vector equation

$$k_1\vec{v_1} + k_2\vec{v_2} + \ldots + k_n\vec{v_n} = \vec{0}$$

and see if $k_1 = 0,\ k_2 = 0,\ \ldots,\ k_n = 0$ is the only solution or if there are other *(infinitely many)* solutions.

$\hookrightarrow$ EXAMPLE 5: Confirm that the three eigenvectors obtained in EXAMPLE 4, $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are really linearly independent.

$\hookrightarrow$ ANSWERS: We seek scalars k_1, k_2, and k_3 such that $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$, or equivalently

$$\begin{bmatrix} -2k_1 - k_2 \\ k_1 \quad + k_3 \\ k_1 + k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\rightsquigarrow$ NOTE the coefficient matrix above in red is simply made up of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ as column vectors. This will *ALWAYS* be the case when solving an equation of the form Equation (2) for $k_1, \ldots, k_n$.

Using Gaussian elimination on this system, we do $R1 \longleftrightarrow R3$ THEN $R2 \mapsto R2 - R1$ and $R3 \mapsto R3 + 2R1$ to get the equivalent system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{REMINDER} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Next, $R3 \mapsto R3 + R2$ leads to

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

With the system now in upper-triangular form, we see that the only solution is $k_1 = 0$, $k_2 = 0$, and $k_3 = 0$.

So, the three eigenvectors

$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ from EXAMPLE 4

ARE linearly independent as claimed earlier.

$\hookrightarrow$ EXAMPLE 6: Are the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ linearly independent or linearly dependent?

$\hookrightarrow$ ANSWER: As we saw in the previous example, this amounts to solving the system $A\vec{k} = \vec{0}$, where the columns of A are simply $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$:

$$\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R3 \mapsto R3 - 3R1$ and $R2 \mapsto R2 + 2R1$ leads to the equivalent system

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & -16 & -8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \underset{R3 \mapsto R3 + R2}{\sim} \begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The second line says $\boxed{k_2 = -\frac{1}{2}k_3}$ and the first line says

$k_1 = -5k_2 - 3k_3 = -5(-\frac{1}{2}k_3) - 3k_3 \Rightarrow \boxed{k_1 = -\frac{1}{2}k_3}$.

So a typical solution vector is

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}k_3 \\ -\frac{1}{2}k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

↪ So there are infinitely many solutions. This is all that's important - that $k_1 = 0, \; k_2 = 0, \text{ and } k_3 = 0$ is NOT the only solution. Hence the given vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ are *linearly dependent*.

↪ $\boxed{\text{ALTERNATIVE TESTS FOR LINEAR INDEPENDENCE}}$ Recall that the following statements are equivalent *(i.e., if one is true the others are true and if one is false the others are false)*: **[1]** $\det(A) \neq 0$, **[2]** $A\vec{x} = \vec{b}$ always has a unique solution $\vec{x}$ for given A and $\vec{b}$, and **[3]** A is invertible (or non-singular). Thus another way to test whether a set of n vectors S in $\mathbb{R}^n$ (or more generally in an n-dimensional vector space) is linearly independent is to form the matrix A whose columns *(or rows)* are the n vectors in S. If A is invertible or, equivalently, $\mathbf{det(A) \neq 0}$ *(probably the easiest to check)* or, equivalently, $A\vec{x} = \vec{b}$ has only one solution vector for any $\vec{b}$, then the set of vectors S is *linearly independent*. Otherwise, it is linearly dependent.

NOTE

In this **Applications** section, the only part that is important for **MATH1134** is the subsection on *diagonalisation of a matrix*.

Powers of a Matrix Part 1

- If you know $A\vec{x} = \lambda\vec{x}$, observe that multiplying both sides of the equation on the left by A (and using the laws of matrix algebra) leads to:

  $$A(A\vec{x}) = A(\lambda\vec{x}) \quad \Rightarrow \quad A^2\vec{x} = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}.$$

  Similarly, multiplying both sides of $A^2\vec{x} = \lambda^2\vec{x}$ on the left by A leads to $\mathbf{A^3\vec{x} = \boldsymbol{\lambda}^3\vec{x}}$. In general,

  $$A^n\vec{x} = \lambda^n\vec{x}, \; n = 1, 2, 3, 4, \ldots,$$

  where A has eigenvalue λ with corresponding eigenvector $\vec{x}$.
- Another way to think about this is as follows: *If A has eigenvalue λ with corresponding eigenvector $\vec{x}$, then A^n has eigenvalue λ^n with the SAME corresponding eigenvector $\vec{x}$.*

$\hookrightarrow$ EXAMPLE 7: Recalling from EXAMPLE 4 that

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

find $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}^5 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ without doing any matrix-vector multiplications.

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}^5 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 2^5 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -32 \\ 0 \\ 32 \end{bmatrix}.$$

You can also try this the long way to confirm you have the correct answer!

Diagonalisation of a Square Matrix

- KEY DEFINITIONS: A square matrix A is **diagonalisable** if there exists an *invertible* matrix P such that $\mathbf{P^{-1}AP}$ is a diagonal matrix (call it D). The matrix P is said to **diagonalise** A.
- KEY RESULT: $A_{n\times n}$ is *diagonalisable* if and only if A has **n linearly independent eigenvectors**. *(So not all matrices are diagonalisable - see examples later).*
- HOW TO DIAGONALISE A MATRIX $A_{n\times n}$:
  1. Find n linearly independent eigenvectors for A, $\vec{p}_1,\ \vec{p}_2,\ \ldots,\ \vec{p}_n$.
  2. Form the matrix P with $\vec{p}_1,\ \vec{p}_2,\ \ldots,\ \vec{p}_n$ as its columns.
  3. The matrix $P^{-1}AP$ will be diagonal, of the form

     $$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & \lambda_n \end{bmatrix},$$ where λ_i is the *eigenvalue* of A corresponding to *eigenvector* $\vec{p}_i$ $(i = 1, 2, \ldots, n)$.

- Of course, step 1 in the method might not be possible - in which case the matrix is not diagonalisable. It will be obvious while looking for eigenvalues and eigenvectors of an $n \times n$ matrix when it does **NOT** have n linearly independent eigenvectors.

- $\boxed{\text{EXAMPLE 8:}}$ Recalling from EXAMPLE 2 that $A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$ has eigenvalue $\lambda_1 = 7$ with corresponding eigenvector $\vec{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and eigenvalue $\lambda_2 = -4$ with corresponding eigenvector $\vec{p}_2 = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$, then a matrix P which diagonalises A is

$$P = \begin{bmatrix} 1 & -5 \\ 1 & 6 \end{bmatrix} \quad \left(\text{or } P = \begin{bmatrix} -5 & 1 \\ 6 & 1 \end{bmatrix}\right).$$

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 6 & 5 \\ -1 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 6 & 5 \\ -1 & 1 \end{bmatrix} \quad \left(\text{or } P^{-1} = -\frac{1}{11} \begin{bmatrix} 1 & -1 \\ -6 & -5 \end{bmatrix}\right).$$

So we expect $P^{-1}AP = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \quad \left(\text{or } P^{-1}AP = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix}\right).$

CHECK:

$$P^{-1}AP = \tfrac{1}{11}\begin{bmatrix} 6 & 5 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 1 & 6 \end{bmatrix} =$$

$$\frac{1}{11}\begin{bmatrix} 42 & 35 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 1 & 6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 77 & 0 \\ 0 & -44 \end{bmatrix}$$

$$\mathbf{=}\begin{bmatrix} \mathbf{7} & \mathbf{0} \\ \mathbf{0} & \mathbf{-4} \end{bmatrix} \quad \textbf{as expected.}$$

I'll leave you to check the other case, in which we change the order of the columns of P:

$$P^{-1}AP = -\frac{1}{11}\begin{bmatrix} 1 & -1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix}.$$

- KEY RESULT An $n \times n$ matrix with **n different** eigenvalues is diagonalisable *(since it is guaranteed to have n linearly independent eigenvectors)*. Problems only arise if the matrix has one or more *repeated* eigenvalue(s). In that case, it might or might not be diagonalisable . *(See the following 2 examples)*.

$\hookrightarrow$ $\boxed{\text{EXAMPLE 9:}}$ From EXAMPLE 4 with $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ we found eigenvalues $\lambda_1 = 1$ with corresponding eigenvector $\vec{p}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and eigenvalue $\lambda_2 = 2$ of multiplicity 2 with two corresponding *linearly independent* eignvectors $\vec{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

So with $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ we expect $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Check $P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ and

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -1 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$\hookrightarrow$ $\boxed{\text{EXAMPLE 10:}}$ Is the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ diagonalisable? If so, diagonalise it.

$\hookrightarrow$ $\boxed{\text{ANSWER:}}$ First we find the eigenvalues: we solve $\det(A - \lambda I) = 0 \Rightarrow$

$$\begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (3-\lambda)^2(4-\lambda) = 0$$

so the eigenvalues are $\lambda_1 = 3$ *(of multiplicity 2)* and $\lambda_2 = 4$.

- We need only investigate the eigenvector(s) of $\lambda_1 = 3$ to see if the matrix is diagonalisable *(WHY?)*
  So we seek $\vec{x} = [x_1, x_2, x_3]^T$ such that $(A - 3\mathrm{I})\vec{x} = \vec{0}$, or equivalently

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

REMINDER: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- This system is already in (upper) triangular form so we can solve directly:
- The first row states that $\boxed{x_2 = 0}$.
- The second row states that $0 = 0$ and
- The third row states that $\boxed{x_3 = 0}$.

So a typical eigenvector corresponding to eigenvalue $\lambda_1 = 3$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence there is *only one* family of eigenvectors and we cannot find 2 **linearly independent** eigenvectors corresponding to eigenvalue $\lambda_1 = 3$ *(compare to EXAMPLE 9)*. Therefore A is NOT DIAGONALISABLE (it will not have 3 linearly independent eigenvectors).

Powers of a Matrix Part 2

- First observe that if D is an $n \times n$ *diagonal* matrix with its only non-zero entries being the diagonal entries d_{11}, d_{22}, ..., d_{nn}, then for any positive integer $k = 1, 2, 3, \ldots$, D^k is also a diagonal matrix with its only non-zero entries being the diagonal entries d_{11}^k, d_{22}^k, ..., d_{nn}^k *(in that order)*.

- SKETCH OF PROOF: I will show the result for D^2 and it will then be obvious that the same argument works for $D(D^2) = D^3$, and $D(D^3) = D^4$ and so on.

  - Recall that $\mathbf{(D^2)_{ij} = \sum_{k=1}^{n} D_{ik} D_{kj}}$. Furthermore, recall $D_{ik} = 0$ if $i \neq k$. Likewise $D_{kj} = 0$ if $k \neq j$. Therefore, the only time that $\sum_{k=1}^{n} D_{ik} D_{kj}$ could involve non-zero terms is when $\boxed{i = k = j}$. So we can first conclude that if $i \neq j$, $(D^2)_{ij} = 0$ so that $\mathbf{D^2}$ **is a diagonal matrix**.

  - Next, if $i = j$ then, recalling $D_{ik} D_{kj} = 0$ unless we also have $k = i$ $(= j)$, then the expression above for $(D^2)_{ii}$ can be simplified as follows:

$$(D^2)_{ii} = \sum_{k=1}^{n} D_{ik} D_{ki} = D_{ii} D_{ii} = D_{ii}^2 \ \text{ or, equivalently } \ d_{ii}^2 \ \text{ as expected.}$$

- Next, if A has been diagonalised by P, so that $D = P^{-1}AP$, observe that $\boxed{A = PDP^{-1}}$. Furthermore

$$\mathbf{A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PDDP^{-1} = PD^2P^{-1}}.$$

- Similarly,

$$\mathbf{A^3 = A(A^2) = PDP^{-1}(PD^2P^{-1}) = PD(P^{-1}P)D^2P^{-1} = \ldots = PD^3P^{-1}}$$

and

$$\mathbf{A^4 = A(A^3) = PDP^{-1}(PD^3P^{-1}) = PD(P^{-1}P)D^3P^{-1} = \ldots = PD^4P^{-1}}$$

and so on ....

- $\boxed{\text{KEY RESULT}}$ Generally, if P diagonalises A, so that $\mathbf{D = P^{-1}AP}$ is a diagonal matrix, then for any positive integer $k = 1, 2, 3, \ldots$,

$$\mathbf{A^k = PD^kP^{-1}}.$$

$\hookrightarrow$ $\boxed{\text{EXAMPLE 11:}}$ Recall from EXAMPLE 9 that for matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ we have matrices $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ such that $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Check that $A = PDP^{-1}$ and then calculate A^5.

$$PDP^{-1} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \quad \text{as expected.}$$

REMINDER:

$$A = PDP^{-1} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Meanwhile, $A^5 = PD^5P^{-1} =$

$$\begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -2 & -32 & 0 \\ 1 & 0 & 32 \\ 1 & 32 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{-30} & \mathbf{0} & \mathbf{-62} \\ \mathbf{31} & \mathbf{32} & \mathbf{31} \\ \mathbf{31} & \mathbf{0} & \mathbf{63} \end{bmatrix} \quad \textbf{(CHECK!)}$$

Matrix/Vector Functions of a Single Variable

$\hookrightarrow$ Recall that we can have vector functions of single variable, t (for example). Such as

$$\vec{r}(t) = (t^3 - 4t)\vec{i} + \sin t\vec{j} + e^{2t}\vec{k} = \begin{bmatrix} t^3 - 4t \\ \sin t \\ e^{2t} \end{bmatrix}.$$

$\hookrightarrow$ We then differentiate or integrate such vector functions by differentiating or integrating each term individually:

$$\vec{r}'(t) = \dot{\vec{r}}(t) = \begin{bmatrix} 3t^2 - 4 \\ \cos t \\ 2e^{2t} \end{bmatrix}.$$

$$\int \vec{r}(t)\,dt = \begin{bmatrix} \int (t^3 - 4t)\,dt \\ \int \sin t\,dt \\ \int e^{2t}\,dt \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t^4 - 2t^2 + C_1 \\ -\cos t + C_2 \\ \frac{1}{2}e^{2t} + C_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t^4 - 2t^2 \\ -\cos t \\ \frac{1}{2}e^{2t} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

where C_1, C_2, and C_3 are constants.

$\hookrightarrow$ And the old familiar rules from differentiating/integrating of simple functions carry over to differentiating/integrating of vector functions. For example, both operations are linear with respect to *scalar multiplication* and *vector addition*. So

$$\frac{d}{dt}\left(\alpha\vec{v}(t)\right) = \alpha\frac{d}{dt}\left(\vec{v}(t)\right), \qquad \frac{d}{dt}\left(\vec{a}(t)+\vec{b}(t)\right) = \frac{d}{dt}\left(\vec{a}(t)\right) + \frac{d}{dt}\left(\vec{b}(t)\right);$$

and

$$\int\left(\alpha\vec{v}(t)\right)\,dt = \alpha\int\left(\vec{v}(t)\right)\,dt, \quad \int\left(\vec{a}(t)+\vec{b}(t)\right)\,dt = \int\left(\vec{a}(t)\right)\,dt + \int\left(\vec{b}(t)\right)\,dt;$$

for α a scalar.

Other (somewhat) familiar rules are

- $\frac{d}{dt}\left(\vec{a}(t)\cdot\vec{b}(t)\right) = \frac{d}{dt}(\vec{a}(t))\cdot\vec{b}(t) + \vec{a}(t)\cdot\frac{d}{dt}(\vec{b}(t))$.
- $\frac{d}{dt}\left(\vec{a}(t)\times\vec{b}(t)\right) = \frac{d}{dt}(\vec{a}(t))\times\vec{b}(t) + \vec{a}(t)\times\frac{d}{dt}(\vec{b}(t))$.
- $\frac{d}{dt}\left(f(t)\vec{a}(t)\right) = \frac{d}{dt}(f(t))\vec{a}(t) + f(t)\frac{d}{dt}\left(\vec{a}(t)\right)$, where $f(t)$ is a normal (scalar) function.

$\hookrightarrow$ Everything said previously about *vector functions* generalises in the natural way to more general *matrix functions* (except for the results involving the dot product and cross product, which are not defined for non-vector matrices). So to differentiate or integrate a matrix $A(t)$, just differentiate or integrate each entry.

- E.g. $A(t) = \begin{bmatrix} 4t^3 + 2t^2 & \cos 5t \\ 4 & 3e^{12t} \end{bmatrix}$, then

$$\frac{d}{dt}(A(t)) = A'(t) = \begin{bmatrix} 12t^2 + 4t & -5\sin 5t \\ 0 & 36e^{12t} \end{bmatrix}, \qquad \text{and}$$

$$\int A(t)\, dt = \begin{bmatrix} t^4 + \frac{2}{3}t^3 & \frac{1}{5}\sin 5t \\ 4t & \frac{1}{4}e^{12t} \end{bmatrix} + \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

where C_1, C_2, C_3, and C_4 are constants.

- We again, of course, have linearity of matrix differentiation and integration with respect to *scalar multiplication* and *matrix addition*, but also with respect to matrix multiplication by a constant matrix. If C is an appropriately-sized (constant) matrix,

$$\frac{d}{dt}\left(CA(t)\right) = C\frac{d}{dt}\left(A(t)\right) \quad \text{and} \quad \int \left(CA(t)\right)\, dt = C\int \left(A(t)\right)\, dt.$$

- And again with regard to differentiation and *matrix multiplication*, matrix functions mimic scalar functions:

$$\frac{d}{dt}\left(A(t)B(t)\right) = A'(t)B(t) + A(t)B'(t).$$

$\rightsquigarrow$ *NOTE the order of* $A(t)$ *and* $B(t)$ *in the formula above is important since matrix multiplication is NOT commutative.*